Proof that properly discounted present values of assets vibrate randomly

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Even the best investors seem to find it hard to do better than the comprehensive common-stock averages, or better on the average than random selection among stocks of comparable variability. Examination of historical samples of percentage changes in a stock's price show that, when these relative price changes are properly adjusted for expected dividends paid out, they are more or less indistinguishable from white noise; or, at the least, their expected percentage movements constitute a driftless random walk (or a random walk with mean drift specifiable in terms of an interest factor appropriate to the stock's variability or riskiness). The present contribution shows that such observable patterns can be deduced rigorously from a model which hypothesizes that a stock's present price is set at the expected discounted value of its future dividends, where the future dividends are supposed to be random variables generated according to any general (but known) stochastic process. This fundamental theorem follows by an easy superposition applied to the 1965 Samuelson theorem that properly anticipated futures prices fluctuate randomly—i.e., constitute a martingale sequence, or a generalized martingale with specifiable mean drift. Examples demonstrate that even when the economy is not free to wander randomly, intelligent speculation is able to whiten the spectrum of observed stock-price changes. A subset of investors might have better information or modes of analysis and get above average gains in the random-walk model; and the model's underlying probabilities could be shaped by fundamentalists' economic forces.

1. Review

■ Consider a random vector sequence: $\ldots, X_t, X_{t+1}, \ldots, X_{t+T}, \ldots$. The dividend of a particular common stock, say General Motors, might be the ith component of that vector: $\ldots, x_{it}, \ldots, x_{i\,t+T}, \ldots$; and the jth component might, as in 1965 Samuelson,[1] denote the price of spot wheat at time t. Under some known stochastic process generating the random variables, there will be defined basic conditional probabilities

$$\text{Prob}\{X_{t+T} \leq x_{t+T} \mid X_t = x_t, X_{t-1} = x_{t-1}, \ldots\} = P_T(x_{t+T}; x_t, x_{t-1}, \ldots; t) \quad (1)$$

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The author owes thanks to the National Science Foundation for financial aid, and to Kathryn Kaepplein for valuable assistance.

[1] See [3].

and conditional expected values

$$E\{X_{t+T} \mid X_t = x_t,\ X_{t-1} = x_{t-1},\ \ldots\} = {}_{t+T}Y_t$$

$$= \int_{-\infty}^{\infty} x P_T(dx;\ x_t,\ x_{t-1},\ \ldots;\ t)$$

$$= {}_{t+T}F_t(x_t,\ x_{t-1},\ \ldots) \qquad (2)$$

$$E\{{}_{t+T}Y_{t+1} \mid X_t = x_t,\ X_{t-1} = x_{t-1},\ \ldots\}$$

$$= \int_{-\infty}^{\infty} {}_{t+T}F_{t+1}(x_{t+1},\ x_t,\ \ldots)P_1(dx_{t+1};\ x_t,\ x_{t-1},\ \ldots;\ t)$$

$$= E\{{}_{t+T}Y_{t+1} \mid {}_{t+T}Y_t\} \text{ for short.} \qquad (3)$$

Here a Stieltjes integral is written as $\int_{-\infty}^{\infty} f(x)g(dx)$; and when x is a vector, a *multiple* Stieltjes integral is written as $\int_{-\infty}^{\infty} f(x)g(dx)$.

The two basic 1965 theorems can now be recapitulated.

Theorem 1: For $\tau > t$, the sequence $({}_\tau Y_t,\ {}_\tau Y_{t+1},\ \ldots)$ has the martingale property

$$E\{{}_\tau Y_{t+k} \mid {}_\tau Y_t\} \equiv {}_\tau Y_t, \quad (k = 1, 2, \ldots, \tau - 1). \qquad (4)$$

Theorem 2: For the "discounted" sequence,

$${}_\tau Z_t \equiv {}_\tau Y_t / \prod_{j=1}^{\tau-t} \lambda_{t+j}$$

$$E\{{}_\tau Z_{t+1} \mid {}_\tau Z_t\} = \lambda_{t+1}\, {}_\tau Z_t \qquad (5)$$

$$E\{Z_{t+k} \mid {}_\tau Z_t\} = \lambda_{t+1}\lambda_{t+2} \ldots \lambda_{t+k}\, {}_\tau Z_t.$$

2. Expected present discounted values

■ Suppose that the ith component of the vector X_t represents the dividend of a given stock that is to be paid out at time t. Then if $\lambda_{t+1} - 1$ is the interest rate paid at the end of period t on each dollar invested at time t, and if x_{it} were a nonrandom sequence, the classical Fisher present discounted-value rule of capitalization (slightly generalized) defines the value of a stock as

$$V_t = \sum_{T=1}^{\infty} \left(x_{i\,t+T} / \prod_{j=1}^{T} \lambda_{t+j}\right) \qquad (6)$$

$$V_{t+1} = \lambda_{t+1} V_t - x_{i\,t+1}. \qquad (7)$$

If $\lambda_t \equiv 1 + r$, the above denominator takes on the more familiar form $(1 + r)^T$.

But now revert to the supposition that $x_{i\,t+T}$, and hence V_t, are random variables; and assume that *the market capitalizes the stock at the expected value of* V_t, namely at v_t defined by

$$v_t = E\{V_t \mid X_t = x_t,\ X_{t-1} = x_{t-1},\ \ldots\} = \sum_{T=1}^{\infty} {}_{t+T}Z_t \qquad (8)$$

$$E\{v_{t+1} \mid v_t\} = \sum_{T=2}^{\infty} E\{{}_{t+T}Z_{t+1} \mid {}_{t+T}Z_t\}. \qquad (9)$$

Now, by simple use of the principle of superposition, we can derive from (5) our needed generalization or corollary of Theorem 2, namely that stock prices themselves have a martingale or random-walk property.

Theorem 3. If stocks are capitalized at their expected present discounted values defined by (8) and (9), then

$$E\{v_{t+1}|v_t\} = \lambda_{t+1}v_t - E\{x_{i\ t+1}|X_t = x_t, X_{t-1} = x_{t-1}, \ldots\}. \quad (10)$$

Clearly (10) is the fundamental stochastic generalization of the fundamental nonstochastic relation (7). Note that it holds even for the Pareto-Lévy distributions that lack a finite variance but possess a defined first moment.

Proof of the theorem follows immediately from substituting Theorem 2's relation (5) into each term of (9) and then identifying what remains by use of (8).

3. Example of Brownian ramble

Suppose that the ratio of dividend to earnings is a constant payout fraction. Let earnings at time t be proportional to a random variable satisfying an independent multiplicative relation. Then we can deduce that dividends will be generated by the stochastic process

$$x_{i\ t+T} = x_{it}Z_1 \ldots Z_T, \quad (11)$$

where the Z's are positive random variables subject to uniform and independent probability distributions

$$\begin{aligned} &\text{Prob}\{Z_i \leq z\} = P(z) && (12)\\ &E\{Z_i\} = \theta,\ E\{x_{i\ t+T}\} = \theta^T x_{it} \\ &E\{\log Z_i\} = \mu < \log\theta,\ \text{Var}\{\log Z_i\} = \sigma^2. \end{aligned}$$

Finally, assume a constant interest rate, $\lambda_t \equiv 1 + r > \theta$, which is large enough to keep v_t a finite converging series

$$v_t = x_{it}\left[\frac{\theta}{1+r} + \frac{\theta^2}{(1+r)^2} + \cdots\right] = x_{it}\theta(1 + r - \theta)^{-1} \quad (13)$$

$$\begin{aligned} &\text{Prob}\{v_{t+1}/v_t \leq z\} = P(z) && (14)\\ &E\{v_{t+1}|v_t\} = \theta v_t \quad \text{from (12)} \\ &\qquad = (1+r)v_t - E\{x_{i\ t+1}\} \quad \text{from (10).} \end{aligned}$$

Actually this model generates the economic or multiplicative Brownian motion of Osborne and Samuelson[2] with the asymptotic log-normal distribution

$$\lim_{T\to\infty} \text{Prob}\left\{a \leq \frac{\log(v_{t+T}/v_t) - \mu T}{T^{\frac{1}{2}}\sigma} \leq b\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}s^2}ds \quad (15)$$

and its price changes have the white-noise property

$$E\{\log v_{t+1} - \log v_t - \mu\} \equiv 0 \quad (16)$$

$$\text{covariance}\{\log v_{t+T}, \log v_t\} \equiv 0, \quad T > 0. \quad (17)$$

Granger[3] has arrived at similar results, including the interesting case where variables are generated as the (possibly infinite) sum of

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[2] See [2] and [4], respectively.

[3] In [1].

white-noise random variables. Shiller[4] also offers valuable related contributions, particularly in connection with prediction algorithms and also the term structure of interest rates.

4. Probabilities that obey economic law

■ A second model provides an interesting contrast to the endless wandering of the above model. In it, earnings and dividends continue to have a probability distribution that stays within the same general central range; thus dividends have an ergodic distribution that is determined by economic law, by the fundamentals of the industry's resource scarcities and the capacity of its goods to meet peoples' needs and demands. But, and this is the beauty of the present martingale process, the movement of the stock price that capitalizes these determinate dividends is itself a white-noise generalized martingale!

Specifically, let dividends satisfy a damped autoregressive process

$$\log x_{i\,t+1} = a \log x_{it} + \eta_t, \quad |a| < 1, \tag{18}$$

where η_t is an independently and uniformly distributed random variable, with cov $(\eta_t, \eta_{t\pm k}) = 0$ for $k \neq 0$.

Then, for $|a| < 1$,

$$\lim_{T\to\infty} \operatorname{Prob}\{x_{i\,t+T} \leq x \,|\, x_{it} = y\} = \lim_{T\to\infty} P_T(x, y) = P(x), \tag{19}$$

a limiting ergodic probability distribution that is independent of initial value for x_{it} and which is not log-normal.

Even though dividends and their changes have a nonwhite spectrum, with nonvanishing covariance $\{x_{it}, x_{i\,t\pm k}\}$, the martingale property of Theorem 3's (10) will still be valid. Thus, if the corporation had zero dividend payments over a time interval, and the λ_{t+j} discount factor were at or near unity, the spectrum of $v_{t+k} - v_t$ would be white, in the sense of zero first-order autocorrelation and zero expected values.

The present case of an ergodic probability distribution differs significantly from the log-normal models upon which so much of warrant and option valuations has been based. As applied to calls, which are typically warrants *protected* for dividend payouts, the difference is not so great. Indeed, as my colleague Robert Merton reminds me, even for the present model, once we ask what will be the cumulative value over time of a portfolio that invests back all dividends in this company's common stock, the relevant probability distribution derived from (19) will have properties much like that of (15). In fact, in the following special case, we shall have exactly the same form as (15).

Suppose the corporation selects its optimal algebraic dividend payout so as to leave within the company only that sum of wealth or money which can optimally earn more there than elsewhere. (If the indicated dividend is negative, think of the corporation as selling new shares; for that matter, transaction costs and tax complications aside, a corporation might choose always simply to buy shares *algebraically* in the open market, so that any positive dividend situation would work itself out in each of my shares' becoming more valuable.) Suppose further, for simplicity, that *ex ante* always the

[4] In [5].

same total wealth is to be left in the company: all the random events of the period just past show up in the variable algebraic dividend. Finally, let the relevant interest rates be constant, $\lambda_t \equiv 1+r$. Then each dollar left invested and reinvested in this company will be subject to the multiplicative probability distribution of (11)'s form; and (15)'s log-normal limit will apply. Even if the amount the company is to reinvest is not completely independent in probability from period to period, the white martingale property assures zero autocorrelation and unbiased means; consequently a slight generalization of the central-limit theorem, to unautocorrelated rather than independent added variates, ought still to enable derivation of a log-normal limit.

5. Qualifications

■ One person, too small to affect market prices appreciably, could make systematic speculative gains in excess of those shown in (10), if he had more or better information or a better way of evaluating existing information. This would enable him to improve upon the probability distribution of (1). Thus, suppose at time t he could know x_{t+1} exactly, or have a more accurate way of estimating it than from $P_1(x_{t+1}; x_t, x_{t+1}, \ldots; t)$.

An example would be where this investor had private knowledge, or private recognition, of an additional datum m_t, in terms of which he has the probability distribution $Q_1(x_{t+1}; x_t, x_{t-1}, \ldots; m_t; t)$ with the property that $P_1(x_{t+1}; x_t, x_{t-1}, \ldots; t)$ is the "marginal distribution" of $Q_1(\ \ )$ with m_t integrated out. Suppose

$$P_1(x_{t+1}; x_t, x_{t-1}, \ldots; t) \equiv \int_{-\infty}^{\infty} Q_1(x_{t+1}; x_t, x_{t-1}, \ldots; dm_t; t), \quad (20)$$

and

$$Q_1(x_{t+1}; x_t, x_{t-1}, \ldots; m_t; t) P_1(x_{t+1}; x_t, x_{t-1}, \ldots; t)^{-1} \neq \text{a function of } m_t \text{ alone.}$$

Then knowledge of m_t gives extra predictive power of $x_{i\ t+T}$ and of V_{t+k}. Having such knowledge when others do not is highly profitable, since depending upon the level of m_t, the stock becomes an especially good or an especially bad buy. Of course, if this private knowledge becomes widespread, the relevant $P_1(\ )$ will become $Q_1(\ )$ itself, with Theorem 3 and (10) holding in terms of it, and with m_t being just one more element in the relevant x_t. In summary, the present study shows (a) there is no incompatibility in principle between the so-called random-walk model and the fundamentalists' model, and (b) there is no incompatibility in principle between behavior of stocks' prices that behave like random walk at the same time that there exist subsets of investors who can do systematically better than the average investors.

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