Chapter One

NOISY RATIONAL EXPECTATIONS WITH STOCHASTIC FUNDAMENTALS

1 Introduction

Models of trading behavior often use the assumption of rational expectations to describe how traders form beliefs about the value of assets. In this way, beliefs are allowed to be formed endogenously. Rational expectations trading models specify the information which is available to traders, and assume this information is used rationally to form beliefs. Information may be in the form of market signals as to the value of an asset, and these signals may be drawn from a distribution known to each trader. In this situation, a rational expectations trading model can be used to describe the formation of beliefs, and the convergence or nonconvergence of beliefs as to the value of an asset. Rational expectation models are important not only to explain trading activities of market participants, but also may be applied to the understanding of price formation in general. For example, these models may give insight into how an exchange rate might be related to fundamental macroeconomic factors in an economy such as price indices.

In this chapter the model formalized by Grossman & Stiglitz (1980) and modified by Blume, Easley and O'Hara (1994) is extended to allow the true value of the risky asset to be determined by an exogenous fundamental. This fundamental follows a discrete-time continuous-state random walk rather than remain constant throughout the trading sessions. Agents receive signals as to the value of the asset as in previous versions of this model, and these signals are allowed to converge to the true value of the asset. The equilibrium is a fulfilled expectations equilibria in the sense of Kreps (1977) in that when changes to the fundamental are less than fully revealing, the expected changes in the value of the fundamental influences the price of the risky asset.

Of primary interest is how a non-stationary component of the price process changes the volume-signal precision relation. In the work of Blume, Easley and O'Hara, volume was used to resolve the uncertainty of the uninformed traders as to the signal received by informed traders. It is shown that for their results to hold in a multi period setting, the underlying fundamental in the model must follow a stochastic process.

The analysis will be presented in four remaining sections. Section 2 describes Grossman & Stiglitz type generalized noisy rational expectation models with supply uncertainty. In Section 3, the extensions of Blume, Easley and O'Hara are presented along with their conclusions regarding the role of volume in the price process. New results are derived in Section 4 where the value of the risky asset follows a simple discrete-time continuous-state process. The role of volume is then reinterpreted under these specifications. Section 5 summarizes the conclusions of the chapter.

2 Noisy Rational Expectations Models

Grossman and Stiglitz (1980) demonstrate that when information is costly, markets in equilibrium cannot be assumed to be perfectly arbitraged. The decision by traders to be informed is specified as an endogenous variable, and in equilibrium it is shown that the proportion of informed traders can be less than unity. This implies that some traders choose not to be informed. Traders choose not to be informed either because information is too costly or because prices convey information which might otherwise be purchased.

An unanswered question in this model is how information from the informed traders is transmitted through the price system to the uninformed traders. When some portion of traders choose to be informed, a condition for equilibrium is that the ratio of the expected utility of the informed and the uninformed traders is unity. As more traders choose to be informed, the price system is characterized as being more informative. However, when information is costly, the marginal benefit of purchasing additional information will at some point fall short of its cost, and some traders will always choose to remain uninformed. Uninformed traders will depend on the price system alone to determine their demands, and assume informed traders transmit costly information to the price system thereby allowing prices to become informative.

Trading in this model is based on differences in beliefs. Traders demand some quantity of a risky asset based on rational expectations and utility maximization. In the extremes when either all traders choose to be informed or no traders choose to be informed, markets become thin and trading breaks down. This condition occurs when either there is little noise in the system so there is no need to become informed, or when the cost of information is very low so all traders become informed.

The structure of the model is based on the rational expectations models of Radner (1968), Green (1977), and Lucas (1972).[1] There are two assets, a safe asset and a risky asset for which a budget constraint is defined along with initial wealth. The quantity of the risky asset is uncertain. The return on the risky asset is $u = \theta + \varepsilon$, where θ is observable at a cost and (θ, ε) are uncorrelated independent normal zero mean random variables. All traders are identical initially and understand the distribution of returns based on prices. Some choose to be informed where λ is defined as the proportion informed. Each informed trader receives the same signal. For an equilibrium proportion of informed traders, λ, a price function, $P_\lambda(\theta, x)$ is defined where θ is the costly signal, and x is the random quantity supplied of the risky asset.

Traders are risk averse and have identical constant absolute risk aversion utility functions. A negative exponential utility function is used, $V(w) = - e^{-(a w)}$, where w is wealth and a is the measure of risk aversion. Informed traders form demand for the risky asset by maximizing their utility given the current price and signal. With the specified form of the utility function, demand can be solved as, $X(P, \theta) = (\theta - R P) / a \sigma_\varepsilon^2$, where P is the equilibrium price, R is the return on the safe asset, and σ_ε^2 is the variance of the return on the risky asset u given the noisy signal.

Uninformed traders observe price but not the signal observed by the informed traders. It is assumed that after repeated observation, uninformed traders learn the relation between the observed price and the return on the risky asset, and form expectations rationally. The demand of the uninformed is described as a function of the learned

[1] For a discussion of the Green-Lucas theorem see e.g. Laffont (1993).

equilibrium price function and observed price. The price function assumed for the uninformed traders, $P^*(\bullet)$, is based on the unobserved factors (θ, x), and is formed such that u and P^* are jointly normally distributed. That is, uninformed traders form price expectations which relate statistically to the return of the risky asset. The existence of such a price function is proven in Grossman & Stiglitz.

Uninformed traders use this price function along with the currently observed price to formulate their demand for the risky asset as

$$X(P, P^*) = (E[u^*| P] - R P) / (a V[u^*| P]), \tag{1}$$

where $E[\bullet]$ is the mathematical conditional expectations operator, and $V[\bullet]$ is the conditional variance. With the demand of both the informed and uninformed traders, an equilibrium demand *system* is defined. An equilibrium price system, $P_\lambda(\theta, x)$, is a function of (θ, x) such that for (θ, x), the sum of the demand of the informed and uninformed traders equals supply.

For this model to be useful, the equilibrium price system must have a specific form. The equilibrium price system is characterized as a linear function in (θ, x) of a particular form of noisy signal defined as

$$s_\lambda(\theta, x) = \theta - (a\, \sigma^2 / \lambda)\, (x - E[x \mid x^*]), \tag{2}$$

with $\lambda > 0$. The first term on the right hand side is the noisy signal, and the the second term on the right hand side represents noise due to supply uncertainty. With $\lambda = 0$, there is no signal and only the current demand is available. Show this as

$$s_\lambda(\theta, x) = x. \tag{3}$$

Grossman & Stiglitz prove that equilibrium price is in fact a linear function of s_λ and that prices convey information about θ. It is shown that s_λ is a mean-preserving spread of θ, and therefore the equilibrium price system can be characterized as

$$P_\lambda(\theta,x) = \lambda_1 + \lambda_2\, s_\lambda. \tag{4}$$

Information (θ) is transmitted from the informed traders to the uninformed traders by allowing the uninformed to observe P and learn P^*. Since s_λ is a mean-preserving spread of θ, the expected value and variances can be written as

$$E[s_\lambda{}^* \mid \theta] = \theta$$

and (5)

$$V[s_\lambda{}^* \mid \theta] = ((a^2\, \sigma_\varepsilon{}^4) / \lambda^2)\ V[x^*].$$

The conditional variance of $s_\lambda{}^*$ measures how well information from the informed trader is transferred to the uninformed. It depends on the overall noise in the system, the supply noise, the noise in the signal, the coefficient of risk aversion, and is inversely related to the proportion of informed traders. As more traders choose to be informed, more information is transferred to the uninformed traders.

Since the number of informed traders is an endogenous variable, equilibrium must also occur in the information market: The number of traders who wish to become informed must be determined. Information market equilibrium is reached when the ratio of the expected utility of the informed and uninformed traders is unity. An overall equilibrium includes an equilibrium price system and equilibrium in the information market. This equilibrium depends on the cost of information, the quality of the informed traders' information, and the coefficient of risk aversion.

Comparative statics show that as information quality increases, the price system becomes more informative. A decrease in the cost of information or a decrease in risk aversion increases the informativeness of the price system. All other changes in parameters other than these just mentioned do not change the informativeness of the price system but only change the proportion of traders who choose to become informed. For example, an increase in supply noise will first decrease the informativeness of the price system but at the same time more traders will choose to be informed which increases the informativeness of the price system. These effects will cancel leaving the informativeness of the price system unchanged. Finally, as signals becomes more informative, the informativeness of the price system increases. Additional uninformed traders will choose to purchase signals, and a new equilibrium proportion of informed traders will be reached where the expected utility of the informed and uninformed traders is balanced.

The Grossman & Stiglitz framework is an important beginning to an understanding trading in situations of asymmetric information. Soon after the model introduced, however, the assumption of trade with common knowledge was challenged. Milgrom & Stokey (1982) proved that in a rational expectations framework with common knowledge, traders who begin with a Pareto optimal allocation will never agree to trade when new information is available to some traders.[2] Recall that in the Grossman & Stiglitz model trade is based on differences in belief. All informed traders purchase the same signal while the uninformed receive no new information other than the current market price. Following the argument of Milgrom & Stokey, the uninformed traders as a group would refuse to trade with the group of informed traders whenever it is known that the informed traders do

[2] See also Sargent (1993) for discussion of Tirole's (1982) version of the no-trade theorem along with Sargent's alternative approach which employs boundedly rational agents.

in fact have better information.

The no-trade scenario is difficult to reconcile with empirical observations of liquid markets. Hellwig (1980) and Diamond & Verrecchia (1981) argue that prices are at best only partially revealing, and therefore there are always incentives to collect costly information. If this is true then the fully revealing no-trade situation will not occur. This is accomplished by making information dispersed rather than uniform across traders as in the Grossman & Stiglitz model. In small finite economies, when information is dispersed the individual demands of the informed trader cannot reveal all available information. In very large economies, individual traders with dispersed information cannot effect prices if their size is small relative to the size of the market. Prices however can take on the role of aggregators of the dispersed information of informed traders. The idea that prices aggregate information can be traced back to Hayek's (1945) essay, "The use of knowledge in society".

Diamond & Verrecchia (1981) also argue that noise in rational expectations models in the form of supply uncertainty can prevent prices from being fully revealing, while if costly information can be purchased to resolve this uncertainty then as in the Grossman & Stiglitz model, a no-trade situation can occur. However, Diamond & Verrecchia suggest that the no-trade situation does not occur when information is dispersed. That is, when information aggregated across traders is more valuable than the information of a single trader. This implies that private information is always valuable, and aggregated information is more informative than information belonging to any individual trader.

The model of Diamond & Verrecchia may be described as a Bayesian statistical decision problem. Traders learn the behavior of prices and the return on a risky asset, and form demands for the risky asset conditional on their endowment of the risky asset, their private information, and the current price for the risky asset. Traders have identical preferences given by identical negative exponential utility functions, identical priors as to the return of the risky asset, and observe costlessly information about the return of the risky asset. Information is of the same precision across all traders yet is dispersed. That is, information signals are drawn from the same distribution for all traders where the mean of the distribution is the common prior for the return on the risky asset, and the variance of the distribution is constant.

The return on the risky asset is Normally distributed, $u \sim N(y_o, 1/h_o)$, where y_o is the common prior, and h_o is the precision or the inverse of the variance of the distribution. The return on the risky asset along with the signals available to traders, (u, y), form a jointly bivariate Normal distribution with mean (y_o, y_o) and covariance matrix[3]

$$\begin{bmatrix} \frac{1}{h_o} & \frac{1}{h_o} \\ \frac{1}{h_o} & \frac{1}{h_o} + \frac{1}{n} \end{bmatrix} \tag{6}$$

where h_o is defined as above and n is the number of draws from the given distribution.

Noise is introduced as aggregate supply uncertainty. Each trader's endowment of the risky asset is an independent draw from a Normal distribution with zero mean and constant finite variance. The realization of a draw for an individual trader, t, is given by $\tilde{x}_t = x_t$. The sum of all individual endowments is $\tilde{X} = \sum \tilde{x}_t$ where the sum is taken over a finite number, T, of traders. Each trader then uses the information these individual endowments, the costless signal, and available price to form demand equations. The demand equations for each trader can then be written as $D_t(x_t, y_t, P)$ for the risky asset, and $B_t(x_t, y_t, P)$ for the safe asset.

An equilibrium is determined by first conjecturing a price function then showing that the form of the conjectured price function is correct, markets clear, and individual budget constraints are satisfied. Only an outline of the determination of the equilibrium will be given here. The goal is to determine the conditional expectation and conditional variance of the risky asset. Define these as

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[3] For the covariance matrix to have this form where the covariance term is equal to the variance of u, the correlation coefficient for u and y must be equal to unity. This implies a linear relationship between the variables such that $P(u = a + b y) = 1$, for some constants a and b. For a proof, see Rice (1987) p. 126-127.

$$\mu_t = E[u \mid x, y, P] \tag{7}$$

and

$$\sigma^2_t = V[u \mid x, y, P]. \tag{8}$$

Conjecture the price function to be

$$\tilde{P} = \alpha\, y_o + (\beta / T) \sum_{t=1}^{T} \tilde{y}_t - (\gamma / T)\, \tilde{X} \tag{9}$$

where α, β, and γ are coefficients to be determined. Compared to the Grossman & Stiglitz price function, this equation is again linear in the available information while per capita supply appears now as an additional term with a unique coefficient.

By design, the expression, $\tilde{P} - (\alpha + \beta)\, y_o$, is normally distributed with zero mean and with constant precision, H. The expressions for the condition mean and variance of the risky asset can then be determined by defining a vector,

$$Z = [\tilde{u}, \tilde{X}, \tilde{x}_t, \tilde{y}_t, \tilde{P} - (\alpha + \beta)\, y_o] \tag{10}$$

which has a five-variate Normal distribution. This vector can be partitioned as

$$z_1^* = [\tilde{u}, \tilde{X}] \tag{11}$$

and

$$z_2^* = [\tilde{x}_t, \tilde{y}_t, \tilde{P} - (\alpha + \beta)\, y_o]\ . \tag{12}$$

Given that mean and covariance matrix of the multivariate Normal vector Z is known, the conditional distribution vector of z_1^* given z_2^* can be then be found by matrix algebra. Once μ_t and σ^2_t are determined, they may be substituted into the demand equation

$$D_t = (\mu_t - P) / \sigma^2_t \tag{13}$$

and the market clearing condition may be applied

$$X = \sum_{t=1}^{T} D_t = (1 / \sigma_t^2) \sum_{t=1}^{T} \mu_t - T\,P \tag{14}$$

and price may be solved for as

$$P = (1 / T) \left[\sum_{t=1}^{T} \mu_t - X\, \sigma_t^2 \right] . \tag{15}$$

This expression for P in terms of known variables is the of the same form as was conjectured, and therefore yields the equilibrium price and the solution to the model.

The model of Diamond & Verrecchia differs from the Grossman & Stiglitz model in that information here is dispersed rather than asymmetric. In the Grossman & Stiglitz model, prices can fully reveal the information of the informed traders to the uninformed traders. By contrast, when trader behavior is modeled as a Bayesian statistical problem where information is dispersed, prices cannot reveal all aggregate information except in limiting cases. As will be seen, limiting cases are incorporated into studies by Brown &

Jennings (1989), and again by Blume, Easley, and O'Hara (1994).

The Diamond & Verrecchia (1981) model is extended to two periods in Brown & Jennings (1989) where it is shown that past as well as current prices are used by traders to resolve the underlying uncertainty of the model. This is similar to Hellwig (1982), where past prices are used because current prices are not yet available when demands are formed. In Brown & Jennings, uncertainty is again in the form of aggregate supply uncertainty. Traders receive dispersed signals as to the payoff of a risky asset. The demand for this asset in each period is based again on the maximization of a negative exponential utility function conditional on information available to the traders. The overall payoff to each trader after the second period is given by

$$n_o + d_{i\,1} (P_2 - P_1) + d_{i\,2} (u - P_2) \tag{16}$$

where n_o is the quantity of the riskless, non-dividend paying asset held after two periods, $d_{i\,1}$ and $d_{i\,2}$ are the quantities demanded by trader, i, in period 1 and 2 respectively. P_1 and P_2 are the equilibrium prices for each period, and u is the return on of the risky asset.

A rational expectations equilibrium is specified as a pair of demand functions (d_1, d_2) for each trader, and a pair of equilibrium prices (P_1, P_2) such that four conditions are satisfied: 1) Prices are functions of available information through their dependence on demands and supplies. 2) For all possible information sets available to traders, conjectures as to the price function are consistent. 3) Each trader's strategy is feasible and optimal. 4) Markets clear in that

$$x_1 = d_1$$

and (17)

$$x_1 + x_2 = d_2,$$

where x_t are the positive or negative supply increments provided exogenously in each period.

The costless private signals observed by each individual at time, t, are

$$y_{i\,t} = u + \varepsilon_{i\,t} \tag{18}$$

where the error term, $\varepsilon_{i\,t}$ is Normally distributed with zero mean and finite variance. The joint expectation of the error term and the return conditional on available information is zero.

Like the model of Diamond & Verrecchia, this model depends always on limiting cases for its solution. When the Strong Law of Large Numbers is applied to the sum of all individual trader's signal over the total number of traders, an average signal is found. Then, due to the character of the error term on individual signals, this average signal converges to the value of the return on the risky asset. The application of Strong Law of Large Numbers for this situation is

$$Pr\left[\mathop{limit}_{I \to \infty} \sum_{i=1}^{I} y_{i\,t} \;/\, I] \;=\; u\right] = 1\ . \tag{19}$$

where the number of traders, I, is allowed to grow to positive infinity.

A two-period model allows traders to engage in strategic behavior across periods. Brown & Jennings refer to hedging demand the purchase (sale) of units of the risky asset in one period for sale (purchase) in the following period without realizing the payoff from the asset. For this type of activity, traders respond to how asset supplies vary from period-to-period. In each period, asset supply increments are Normally distributed with zero mean and finite variance. Across periods, asset supplies may be correlated where the absolute value of the coefficient of correlation is less than unity and constant through time.

The conjectured prices in each period are interdependent, and are based on the supply of the risky asset and the payoff of the risky asset. Prices in each period are defined as

$$P_1 = \alpha_1 y_o + \beta_1 u - \gamma_1 x_1$$

and (20)

$$P_2 = \alpha_2 y_o + \beta_2 u - \gamma_2 x_1 - \delta_2 x_2.$$

Conjectured prices are identical across traders and linear functions of Normally distributed variables.

A rational expectations equilibrium is shown to exist by maximizing individual traders' demands and applying market clearing conditions, then equating coefficients with the set of simultaneous price equations given above. Existence can only be demonstrated for limiting cases. However, if hedging demand is eliminated, the equilibrium can be more precisely characterized. Brown & Jennings prove that without hedging demand and subject to other parametric restrictions, it is useful in the second period to know the first period price. Knowing previous prices is valuable because it resolves some of the uncertainty in the current period, especially with respect to the signals of the other traders in the market. This is due to the fact that supply increments are cumulative and correlated over time, and prices in each period are less than fully revealing.

A different approach is seen in the two trading models described by Romer (1993) where trading activity in itself causes price movements. Similar to the previous models, the Romer models rely on less than fully revealing prices. However, whereas the Diamond & Verrecchia and Brown & Jennings models employed an information structure where information was dispersed and symmetric across traders, Romer models information as asymmetric across traders. A matrix of typical information structures used in rational expectation trading models is given as Figure I.

In the first Romer model, information is heterogeneous across traders where some traders receive better information than others. Traders initially misweight their own information relative to the market price due to uncertainty. Trading which Romer refers to as "market developments" eventually correct these misweightings as information about other's uncertainty is revealed. Hellwig (1980) describes a similar situation although without the self-correcting mechanism employed by Romer.

In Romer's second model, information is widely dispersed and individuals have little incentive to use their own information. Trading costs change the decision problem for traders and generally inhibit trading. The timing of trades by insiders influence prices and affect asset demands for all traders. With a large number of traders in the market, Romer shows the resulting price movements to be substantial.

These two models are unique in that they employ an asymmetric information structure as was seen in the original Grossman & Stiglitz (1980) model along with dispersed information seen in models following Grossman & Stiglitz. The asymmetry structure results in one group of traders being always better informed than the other traders in the market. Instead of relying on limiting cases for a solution to the model as was done in models following Grossman & Stiglitz, however, Romer has each group of traders conditioning their demands not only on price but also on the actions of other group of traders. This makes the model very difficult to solve unless behavior converges to a strategic equilibrium such as a Nash solution. No such solution is proposed and the models cannot be solved in closed form. Romer uses a multiple dimensional grid search technique to study the behavior of the model.

Until strategic equilibria solutions are developed, limiting solutions to noisy rational expectation models appear to be the most useful for studying trading behavior. The value of limiting solutions to models of dispersed information has been demonstrated by Diamond & Verrecchia (1980), and the two-period extension by Brown & Jennings (1989). In addition, Romer (1993) demonstrates that even with dispersed information, allowing traders to be asymmetrically informed may also explain some types of dynamic price movements. Based on this past research, the most promising new models might incorporate an information structure where traders are asymmetrically informed and possess dispersed information. At present these models must then rely on limiting cases for their solution with an eye towards strategic behavior. The next section presents the model by Blume, Easley and O'Hara (1994) which is the basis for the extensions of the current paper.

Blume, Easley and O'Hara (1994) (hereafter BEO) begin with the assumption that prices are not fully revealing, and some form of technical analysis may be useful to traders. They refer to the model of Brown & Jennings in this regard. The form of technical analysis they study is trading volume. The statistical properties of volume in relation to price have been surveyed by Karpoff (1987), and more recently by Gallant, Rossi, and Tauchen (1991). Volume has been found to be correlated with the absolute value of price changes. BEO attempt to demonstrate how volume may in addition be related to the underlying value of an asset and the dynamic behavior of price movements.[4]

In rational expectations models of trading, noise is often in the form of supply uncertainty. BEO argue that allowing traders to observe volume removes the uncertainty in traditional rational expectation models, and results in equilibria in which prices are fully revealing. This leads to a zero trade result first demonstrated by Grossman & Stiglitz (1980). An additional condition for this result is that volume must be defined as the net demand of profit maximizing risk averse traders. Observed volume may not correspond to this definition if liquidity trading takes place by traders who profit from slowly adjusting prices, or if a subset of traders have inelastic demands for the asset due to exogenous constraints. In the BEO model, volume satisfies this additional condition.

The BEO model presented in the next section differs from previous models in that the supply of the risky asset is fixed, and uncertainty is present only in the value of the information signals given to traders. Information in this model can be characterized as dispersed and asymmetric across traders. The hypothesized role for volume is to resolve the uncertainty due to the asymmetry in the information structure. The model attempts to show that the sequence of past volume and price statistics are informative and welfare

[4] Kreps (1977) also considers the conditions under which traders possess private quantity signals and discusses conditions under which these signals are informative.

improving. In equilibrium, the the results of the model correspond to the observed empirical relation between volume and the absolute value of price changes. The model also allows predictions to be made as to the effect of the quality of information on prices and volume changes.

3 A two-period rational expectations model with asymmetric signal quality

3.1 Description of the model

BEO (1994) employ a finite number of agents, $i = \{1 \ldots I\}$, having identical negative exponential utility functions, and coefficients of risk aversion equal to unity. Agents are endowed with n units of a safe asset that pays a liquidating dividend of one, and zero units of a risky asset which pays an uncertain liquidating dividend known only at the end of the period. All trade takes place among the agents who submit their demands to a Walrasian auctioneer. Since no agents begin with any quantity of the risky asset, it might be assumed that agents submit positive as well as negative claims for the risky asset. Markets clear when net claims are zero, and only the payoff of the risky asset is transferred across agents. End of period wealth is computed as the sum of the safe asset held, and the product of the number of risky assets held and payoff of the risky asset. The are no conditions which prevent negative wealth although some such conditions might be imposed later.

The risky asset's eventual value is Normally distributed. The parameters of the distribution are known to all agents, and account for the agent's priors for the risky asset's payoff. The value of the risky asset is given by

$$\Psi \sim N(\Psi_o, 1/\rho_o) \tag{21}$$

The true value of the risky asset is not known to agents until after trading takes place. Agents, however, do receive signals at the beginning of each trading period which estimate the true value of the risky asset. There are two types of signals drawn from a Normal distribution which differ only in their precision, where precision is defined as the inverse of the variance. The subset of traders receiving the better quality signal are referred to as the informed traders. The remaining traders receive the lesser quality signals and are referred to as uninformed traders. In addition, each signal contains a common Normal mean zero error term.

Traders receive new signals at the start of each trading period. The signals for the informed and uninformed traders are as follows

$$y_{i\,t} = \Psi_o + w_t + e_{i\,t} \tag{22}$$

where $y_{i\,t}$ is the signal for trader i in period t, Ψ_o is the mean value for the distribution in which the true value is drawn, w_t is the common error term defined by $w_{i\ t} \sim N(0, 1/\rho_o)$. The idiosyncratic error term, $e_{i\ t}$ is defined as $e_{i\ t} \sim N(0, 1/\rho_1)$ for the informed traders, and $e_{i\,t} \sim N(0, 1/\rho_2)$ for the uninformed traders. While each group of traders is subject to the variance of the common error term, the informed traders by definition receive signals which have idiosyncratic error terms with less variance with respect to the the uninformed traders. It can be said that informed traders have more precise signals. Alternatively, the informed traders' signals could be said to have greater information content.

The precision of the uninformed traders remains constant over time while the precision of the informed traders signal is a random variable. Although not specified in the BEO paper, this variable may be thought of as being drawn from a uniform distribution defined on $[\rho_2, 1]$. Informed traders know the precision of their signal, and the precision of the signal of the uninformed traders. Uninformed traders know the precision of their own signal but not that of the informed traders. The precision of the common error is common knowledge for both groups of traders, and is constant across trading periods.

The overall precision of each group's signal may be defined as the reciprocal of sum of the common and group specific variances. For group 1 and 2,

$$\rho_t^{si} = \frac{\rho_w \rho_t^i}{\rho_w + \rho_t^i} \qquad i = 1,2 . \tag{23}$$

3.2 Formulation of the Bayesian statistical problem

Similar to Diamond & Verrecchia (1981), BEO present a Bayesian statistical problem which is solved for each group of traders. The return on the risky asset remains constant across periods yet at the start of each period, all traders receive a signal as to its value. It is assumed that Bayesian updating takes place over time to improve the estimation of the value of the risky asset. The choice of distributions from which the signals are drawn makes this a standard problem. The form of each group's set of signals is in the form of the sum of a constant and two Normal mean zero random variables. According to standard statistical theory, this sum can again be considered a draw from a Normal distribution, and each new signal received at the beginning of a trading period can be considered an additional sample from a Normal distribution. If the variance of each period's signal is known, then the family of Normal distributions is then a conjugate family of prior distributions for samples from a Normal distribution. (see, e.g., DeGroot, 1989).

As signals are correlated with the true value of the risky asset, receiving signals is useful in estimating the value of the risky asset. The condition expectation and variance may be defined where each is conditional on the received signal. The posterior of Ψ after each new signal may be represented as the conditional expectation of Ψ where the expectation is taken over all possible values of Ψ. This may be seen as

$$E_t[\psi \mid y_t^i] = \frac{\rho_o \psi_o + \rho_t^{si} y_t^i}{\rho_t^{si} + \rho_o} \qquad i = 1,2 \tag{24}$$

and with the conditional variance is given by

$$V_t[\psi \mid y_t^i] = \frac{1}{\rho_t^{si} + \rho_o} \qquad i = 1,2\ . \tag{25}$$

In addition, each signal can be considered as a draw from a Normal distribution where the mean value is the true value itself and the variance is the updated precision for each type of trader in each period. This may be shown as

$$y^i_t \sim N(\Psi,\ 1/\rho^{si}_t) \qquad i = 1,2. \tag{26}$$

Since distribution of w_t is known and common to both types of traders, define the true value of the risky asset plus the noise due to the common error as the "noisy fundamental". This may be written as

$$\theta_t = \Psi + w_t. \tag{27}$$

Traders never see the true realization of the state because of this noise term, w_t, which is common across both types of traders. Conditional on this noise term, however, the distribution of signals for each group can be rewritten as

$$y^i_t \mid w_t \sim N(\theta_t, 1/\rho^i_t) \qquad i = 1,2 . \tag{28}$$

which indicates that signals are informative as to the noisy fundamental.

While the common error is always present, the idiosyncratic error may be eliminated within a period if the number of traders is very large. This occurs because the idiosyncratic errors across traders are Normal random variables with zero mean and are therefore uncorrelated. After repeated draws these errors will cancel out. More formally, the Strong Law of Large Numbers dictates that a sample mean will always converge in probability of the mean of the distribution from which the sample was taken (see, e.g., DeGroot, 1970). Using the Strong Law of Large Numbers this can be expressed as

$$Pr\left[\underset{n \to \infty}{limit} \overline{y}_n = \theta\right] = 1 \tag{29}$$

where

$$\overline{y_n} = \sum_{i=1}^{n} \frac{1}{n} y^i \quad . \tag{30}$$

The convergence of available information to the value of the return on the risky asset is an integral part of limiting solutions to rational expectations models. Brown & Jennings (1989) use the same technique in the solution to their model. In section 4, however, it will be seen that while signals may converge to a value for the risky asset after sufficient signals have been made available, the asset itself may have already moved away from this value.

3.3 The rational expectations solution

There are several conditions which must be satisfied for a rational expectations solution to be shown. First, a price function is conjectured based on information available to traders. Second, traders maximize a utility function subject to a wealth constraint. Third, it is shown that trade is feasible for all traders, markets clear, and the the conjectured price function is consistent with the result of each trader's utility maximization.

It is assumed traders conjecture a price function for the risky asset which includes their prior for the realization of the risky asset, the average signal received during the period, and a term representing the supply of the asset. The conjectured price function for informed and uninformed traders in the first period is

$$p_t = \alpha \psi_o + \beta \overline{y}_t - \gamma x_t \tag{31}$$

where α, β, and γ are estimated parameters.

In the two-period model of Brown & Jennings (1989), two types of trading are defined: hedging demand referred to the purchase (sale) of units of the risky asset in one period for sale (purchase) in a later period while speculative demand depended solely on the eventual return on the risky asset. Traders are said to be myopic if they engage only in speculative demand where it is assumed that utility is maximized on a period-by-period basis rather than over several periods. The assumption of myopic traders greatly simplified the analysis. Brown & Jennings were able to find closed form solutions to their model only under this assumption of myopic traders.

Demand by myopic traders maximizing a negative exponential utility function is of the same form as in previous rational expectations models. The expected value of the risky asset conditional on available information and the conditional variance are defined above. The price used in the demand equation is the equilibrium price in the current period. The choice of the form of the utility function allows demand to be defined as

$$\delta_t = \frac{E_t[\psi \mid y_t^i] - p_t}{V_t[\psi \mid y_t^i]} \tag{32}$$

The value of the expected value and variance from (24) and (25) can then be substituted into (32),

$$\delta_t = \frac{(1/\rho_t^{si})\psi_o + (1/\rho_o)y_t^i - (1/\rho_o + 1/\rho_t^{si})p_t}{(1/\rho_o)(1/\rho_t^{si})} \tag{33}$$

and simplifying yields

$$\delta_t = \rho_o(\Psi_o - p_t) + \rho^{si}{}_t(y^i{}_t - p_t) \qquad i = 1,2. \tag{34}$$

Imposing a market clearing condition requires that net demand by all traders sum to zero. By summing equation (34) over all traders and setting net demand to zero, the equilibrium price can be found. A full derivation is provided in the Appendix. It will be important for what follows that while net demand is set equal to zero, while the absolute value of demand need not be zero.

The proportion of informed and uninformed traders is an exogenous variable; define μ as the proportion of informed traders. The equilibrium price can then be solved for as a solution to the Bayesian statistical problem for both types of traders where the signals of the informed and uninformed traders are weighted by their respective precisions. This may be shown as

$$p_t = \frac{\rho_o \psi_o + \mu \rho_t^{s1} \overline{y}_t^{\,1} + (1-\mu)\rho_t^{s2} \overline{y}_t^{\,2}}{\rho_o + \mu \rho_t^{s1} + (1-\mu)\rho_t^{s2}} . \tag{35}$$

The expression for the equilibrium price would be of the same form as the conjectured price function if it contained a linear per capita supply term. If we were only interested in what BEO describe as the Walrasian price and demand function of the traders, and supply of the risky asset is in the form of Normal mean zero supply increments, the coefficient on the the per capita supply term could be assumed to be zero. BEO, however, state that their model has an exogenous fixed supply of the risky asset, traders begin with zero units of the risky asset, and all trading is among the agents of the model. The assumption of an exogenous fixed supply rather than supply in terms of zero mean supply increments greatly simplifies the results of the model since the final form of the price function excludes any supply term. Since the goal of the model is to highlight the equilibrium volume effects, fixing supply eliminates confounding volume-supply interactions. Rational expectations models where the form of the distribution of supply increments does affect current prices are described by Walsh (1983).

The solution to each trader's Bayesian decision problem gives an expression for trader's demands but is not sufficient to solve the model. As in previous work, a limiting approach is applied to solve for the equilibrium values of the model. In the limit as the number of signals is allowed to grow very large, the average signal for both types of traders are assumed to each converge to the true value of the return on the risky asset. Since each trader is assigned a unique signal, allowing a large numbers of traders in the economy will produce a large number of signals. By application of the Strong Law of Large Numbers the equilibrium price can be expressed as

$$p_t = \frac{\rho_o \psi_o + (\mu \rho_t^{s1} + (1-\mu)\rho_t^{s2})\theta_t}{\rho_o + \mu \rho_t^{s1} + (1-\mu)\rho_t^{s2}} . \tag{36}$$

This expression for price can now be used to study the characteristics of the equilibrium under various conditions, and specifications for the degree of asymmetry between the two types of traders.

3.4 Characteristics of the equilibrium

Given the information available to the informed traders, knowing the equilibrium price allows them to determine the value of the noisy fundamental by inverting the price function. This is not possible for the uninformed traders because one additional variable is unknown, the precision of the signals of the informed traders. Prices are therefore fully revealing for the informed traders but only partially revealing to the uninformed traders.

Current theory suggests that in situations where prices are only partially revealing, traders often look to other information in the trading environment to improve their estimation of unknown variables. Useful information can include the proportion of informed traders in the market, the distribution of endowments, information gathered from futures or forward markets, or technical factors.[5] BEO focus on trading volume as a technical trading factor, and argue that volume statistics improve the uninformed traders' estimation of the noisy fundamental.

Volume statistics are useful to the uninformed trader if there is a consistent relation between trading volume and the precision of the signal of the informed traders, and this

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[5] In the model of Grossman & Stiglitz (1980), informed traders' demands depend only on θ and p. The uninformed traders' demands depend on p, but the uninformed traders are also expected to learn the relationship between return and price. This relationship is determined by the equilibrium price function. Grossman & Stiglitz show that an equilibrium price function exists for their model, and one such price function depends on λ, the proportion of informed traders in the market. In more complex models, the equilibrium price function may also depend on other useful market information. See also the discussion and references in Chapter 2.

relation can be learned by the uninformed traders. This is true because once the equilibrium price is observed, the only variables unknown to the uninformed traders are the precisions of the informed traders, and the value of the noisy fundamental. If the combination of volume and price reduce the uncertainty as to the value of the precision of the informed trader, then volume statistics have informational value for the uninformed trader.

In empirical work it has been shown there is a statistical relation between volume and the absolute value of price changes (see, e.g., survey by Karpoff (1987)). BEO argue that volume itself is driven in part by the degree of certitude of the agents in their model as to the value of their private information. If the causality could simply be reversed then volume would point directly to precision. The results of BEO demonstrate that while there is not a one-to-one relation between volume and precision, volume tends to increase with precision until the precision of signals reaches the overall noise level of the model ($\rho_1 \leq \rho_W$), and then volume decreases as precision exceeds the noise level of the model ($\rho_1 > \rho_W$). The BEO model also demonstrates the empirical relation between the absolute value of price changes and volume. Combining these two relations then allows uninformed traders to make inferences based on volume. For example, high volume is consistent with large absolute price changes and may indicate low precision for the informed traders.

Unfortunately, there is no simple linear relation between volume and the information held by the informed traders. In fact volume is a proxy for demand only if signed volume[6] is available. Furthermore, uninformed traders are only interested in the quantities demanded by informed traders, and a single volume statistic does not provide this. A single overall demand statistic can mask very heterogeneous behavior by subsets of traders. The most common measure of volume in current use, however, is aggregate non-signed volume. This volume statistic is used in the BEO model to show the relation between volume, prices, and precision.

[6] Signed volume indicates not only the number of transactions but also the direction of each trade. A buy might be recorded as (+1) while a sell may be recorded as (-1). The sum of buys and sells yields the net trade, and is referred to as signed volume.

BEO define per capita volume as an unweighted average of all traders demands for the risky asset.

$$V = \frac{1}{2}\ \frac{1}{N}\ \sum_{i=1}^{N} \left| \, quantity\ demanded^{i} \, \right| \tag{37}$$

To be useful in the current period of trading, the expected volume for that period must be calculated. To find the expectation of the above expression for volume, the presence of the absolute value function must be taken into account. BEO provide a useful lemma in the Appendix of their paper which provides for the expectation of the absolute value of a random variable. A full derivation is provided in the Appendix of this chapter. For any random variable, y ~N(θ , 1/ρ),

$$E[\,|\,\gamma y + a\,|\,] = 2\frac{\gamma}{\sqrt{2\pi\rho}}\exp\left[-\frac{1}{2}\left(\frac{\delta\rho^{1/2}}{\gamma}\right)^{2}\right] + \delta\left[\Phi\left(\frac{\delta\rho^{1/2}}{\gamma}\right) - \Phi\left(\frac{-\delta\rho^{1/2}}{\gamma}\right)\right] \tag{38}$$

for $\delta = a + \gamma\,\theta$ and Φ the cumulative normal distribution.

In a large economy expected per capita volume, v, in period one - given the priors of group one and two and using the price equation to solve for θ − can be expressed using the above lemma as,

$$\begin{aligned} v_1 &= \frac{\mu}{2}\left[\frac{1}{N_I}\sum_{i=1}^{N_I} \left|\, \rho_o(\psi_o - p_1) + \rho_1^{s1}(y_1^i - p_1)\,\right|\right] \\ &+ \frac{(\mu-1)}{2}\left[\frac{1}{N_\mu}\sum_{i=1}^{N_\mu} \left|\, \rho_o(\psi_o - p_1) + \rho_1^{s2}(y_1^i - p_1)\,\right|\right] \end{aligned} \tag{39}$$

rewriting and substituting the expectations operator for its definition

$$
\begin{aligned}
v_1 &= \frac{\mu}{2}\left[E\left|\,\rho_o(\psi_o - p_1) + \rho_1^{s1}(y_1^1 - p_1)\,\right|\right] \\
&\quad + \frac{(\mu-1)}{2}\left[E\left|\,\rho_o(\psi_o - p_1) + \rho_1^{s2}(y_1^2 - p_1)\,\right|\right]
\end{aligned} \tag{40}
$$

and with the help of the above lemma

$$
\begin{aligned}
v_1 = \frac{\mu}{2}&\left[2\,\frac{\rho^{s1}}{\sqrt{\rho_1^1}}\phi\left(\frac{\hat{\delta}_1^1\sqrt{\rho_1^1}}{\rho_1^{s1}}\right) + \hat{\delta}_1^1\left(\Phi\left(\frac{\hat{\delta}_1^1\sqrt{\rho_1^1}}{\rho_1^{s1}}\right) - \Phi\left(\frac{-\hat{\delta}_1^1\sqrt{\rho_1^1}}{\rho_1^{s1}}\right)\right)\right] \\
+ \frac{(1-\mu)}{2}&\left[2\,\frac{\rho^{s2}}{\sqrt{\rho^2}}\phi\left(\frac{\hat{\delta}_1^2\sqrt{\rho_1^2}}{\rho^{s2}}\right) + \hat{\delta}_1^2\left(\Phi\left(\frac{\hat{\delta}_1^2\sqrt{\rho^2}}{\rho^{s2}}\right) - \Phi\left(\frac{-\hat{\delta}_1^2\sqrt{\rho^2}}{\rho^{s2}}\right)\right)\right]
\end{aligned} \tag{41}
$$

where

$$
\hat{\delta}_1^j = \rho_o(p_1 - \psi_o)\left(\frac{\rho_1^{sj}}{\mu\,\rho_1^{s1} + (1-\mu)\rho^{s2}} - 1\right) \tag{42}
$$

is the demand of group j, ϕ is the standard normal density, and Φ the cumulative normal distribution function.

BEO draw three results from this expression for per capita volume. First, under simplifying assumptions, volume is decreasing with the precision of the informed traders. Second, volume is non-linear with respect to price. Third, the limit of volume over time is non-zero.

The first result relies on a comparative static analysis of the expression for per capita volume. Assume for simplicity that the precision of the uninformed trader is very low ($\rho_2 = 0$), then the behavior of volume as the precision of the informed trader increases is given by the derivative of the per capita volume relation with respect to precision. This relation can be shown as

$$\frac{\partial v}{\partial \rho_1^1} = \frac{\mu}{2}\left[\phi\left(\frac{\hat{\delta}_1^1(\rho_w + \rho_1^1)}{\rho_w\sqrt{\rho_1^1}}\right)\left(\frac{\rho_w}{\sqrt{\rho_1^1}}\right)\left(\frac{(\rho_w - \rho_1^1)}{\sqrt{\rho_w + \rho_1^1}}\right)\right] \tag{43}$$

In this expression, μ is always positive, the density of the standard Normal function is always positive, and the ratio of the signal precision to the overall noise is positive. This leaves the last term which depends on the relative magnitude between signal precision and noise. When signal precision is inferior to noise ($\rho_1 \leq \rho_W$), this term causes the entire expression to be positive. As signal precision increases, volume will also tend to increase. When signal precision is greater than the overall noise term, ($\rho_1 > \rho_W$), the entire expression is negative and volume will tend to decrease as signal precision increases. Note also that the slope has the same magnitude except for sign when signal precision is near the level of noise of the system. A maximum for per capita volume occurs when the signal precision is the same as the overall noise level. A graph of these cases is presented as Figure II.

This figure has an intuitive interpretation. While the precision of the informed trader is increasing to the level of noise in the system, more trade occurs as the informed traders are receiving better signals, and are able to act on the information from their signals. Overall, prices are also becoming more informative. Once the precision of the signals surpasses the noise level, however, prices reveal more and more information and volume tends to zero. This is consistent with the no-trade theorems already discussed. The better the quality of signal of the informed trader, the less willing an uninformed trader will be to trade.

The Grossman & Stiglitz (1980) model uses costly information which may be purchased by any trader. As signal quality increases or as prices more completely reveal the noisy fundamental to the informed traders, the dispersion of the expected value of the risky asset decreases until in essence all informed traders have the same estimate. In this way, the BEO result leads to the Grossman & Stiglitz result over time as the informed traders' estimates of the value of the risky asset converge.

The second result describes the relation between price and volume. By taking the derivative of (41) with respect to price, it can be seen that volume increases as the current market price and the traders' prior value for price diverge. BEO use the positive sign of the second derivative is positive as evidence of a convex relation between price and volume. Volume reaches a minimum when price and the prior for price coincide. As volume is defined as always positive, when price does not coincide with the prior, volume increases. This result is supported by simulations of the model for various starting parameters.

The last result describes the behavior of the volume statistic over time. As time proceeds, more information becomes available to all traders since a new signal is received each period. This allows Bayesian updating to be performed each period and after sufficient periods, the true value of the risky asset will be revealed. It would be expected that volume would decline to zero as more information becomes available regarding the true value. BEO argue, however, that this does not occur and the limit of volume after many periods is non zero. They explain that in early periods, traders take limited positions because they are not sure of the true value, whereas in later periods traders take large positions to exploit small price discrepancies.

There are several factors which contribute to this argument. One factor is the lack of a budget constraint in the model. It is assumed that borrowing is unlimited and there is no restriction on the size of positions taken. This allows traders to allocate unlimited resources to exploit very small price deviations. Second, the results of the model are based on uninformed traders having an essentially meaningless signal (a signal with a very large variance). This implies that the uninformed traders are not updating their information other than to adjust their prior. Since prices are not revealing to the uninformed traders, the new value of the prior will not differ significantly from the expected value of the risky asset which is the initial value of the prior. Therefore uninformed traders will not change their behavior over time, and volume over time will not be affected by the uninformed traders. The main contributing factor to the prediction of non zero volume over time is the assumption as to the behavior of the informed traders over time. They argue, "Trade does not disappear because although traders' beliefs are converging to a common belief their precisions are diverging at the same rate," (p. 174). As will be seen below, however, precisions must converge over time if traders are using Bayesian updating, and the importance of any divergent belief to the traders is insignificant.

The time series values for the behavior of volume are constructed by setting the initial prior to the expected value of the risky asset. In subsequent periods, the prior is assigned as the previous period's estimate of the true value of the risky asset as measured by the noisy fundamental. The precision of the uninformed traders is set at zero implying that all signals given to the uninformed trader are meaningless. The precision of the informed trader is selected each period from a Uniform distribution on the range [0,1]. In the first period, the precision of the prior is assigned as the inverse of the variance of the distribution from which the prior was drawn. Over time this prior is updated as new realizations of the noisy fundamental are observed as prices in each period. This allows the noisy fundamental to converge to the true value of the fundamental. Since the updated prior precision increases in each period while the precision of the signals remains constant, at some point the prior becomes more valuable than the individual signal. Bayesian traders recognize this fact and place more and more weight on the updated prior in their demand equation, and less weight on the current period signals. After many periods, the marginal value of a new signal becomes very small because current prices have converged to the true value for the risky asset.

The crucial behavioral assumption as to trader behavior after many periods is that traders continue to act on private signals even when these private signals have almost zero informational value compared with observed price. The argument rests on the fact that since traders continue to receive private signals, prices are not fully revealing. In the limit the weight traders put on their private signal converges to zero but never reaches zero. BEO assume this private signal weight is sufficient to sustain non zero volume.

Fully revealing prices is not an issue with the single period model as prices can never be fully revealing as long as there is only one realization of the noisy fundamental. This is the essential difference between the BEO model and the Grossman & Stiglitz model discussed earlier. When the model is extended to multiple periods, however, prices do become fully revealing for the informed traders if only partially revealing for the uninformed traders. Prices reveal the noisy fundamental in early periods, and in later periods prices reveal the true value as the informed traders use information received in each period to update their estimates and these estimates converge to the true value.

To preserve the BEO results, the convergence of the noisy fundamental to the true value can be prevented by allowing the true value to change over time. The next section shows that in this situation, prices are only partially revealing; and in the presence of asymmetric information, volume is non-zero.

4 Introduction of a stochastic fundamental

The agents of the preceding models had conjectured a price function based on their priors and the current value of the signal. Agents knew that the true value of the risky asset was stationary, and they used Bayesian updating to improve their estimation of the true value of the asset. The resulting price function was linear in the agent's prior and a noisy signal. An alternate approach might alter this price function within the confines of a rational expectations formulation to allow the fundamental to follow a non-stationary process. Assume for the moment that the new price function remains linear in this non-stationary component and the noisy signal as follows:

$$p = \alpha S + \beta \overline{y} - \gamma x \tag{44}$$

where *S* is defined as a fundamental following a discrete-time, continuous-state random walk such that

$$S_t = S_{t-1} + \xi_t \tag{45}$$

with ξ_t, a normally distributed random variable with zero mean. Define the per period variance as $1/\rho_f$. Given the distribution assumption, the expected value for time *t* of this process is S_{t-1},

Informed and Uninformed traders receive a new signals in each period which are related to the fundamental. These signals differ only in their respective error terms. The error term for the informed trader's signal, $e^1{}_t$, is defined to be on average less in absolute value than the error term for the uninformed trader, $\varepsilon^2{}_t$. For the informed trader the signal is

$$y^1{}_t = S_t + w_t + e^1{}_t \tag{46}$$

and the uninformed signal is

$$y^2{}_t = S_t + w_t + \varepsilon^2{}_t \tag{47}$$

As in section 3.1, both types of signals contain noise terms which prevent traders from knowing the true value of the fundamental from a single signal. In the Blume, Easley, and O'Hara model, the fundamental is constant and from period-to-period, traders are able to improve their estimate of the value of the fundamental over time. Here, however, the value of the fundamental changes each period, and due to the definition of fundamental in equation (45), the sequence of previous realizations of the fundamental are not useful.

Traders do observe the previous period's price, and since it is conjectured that price and the fundamental are related as seen in the price equation (44), the most recent price, p_{t-1}, may be useful whereas historical prices are not. Traders would still need to know how price relates to the fundamental. There may be many rational expectations solutions to this price-fundamental relation.[7] The solution considered in this model is that each trader's best unbiased estimate of the fundamental in any period will be the most recent price. The most recent price, p_{t-1}, will then be the prior for each trader in both groups.

Using this new definition of the prior, the expected value of the fundamental conditional on this signal will be

$$E_t[S_t \mid y_t^i] = \frac{\rho_f p_{t-1} + \rho_t^{si} y_t^i}{\rho_f + \rho_t^{si}} \qquad i = 1,2 \tag{48}$$

while the conditional variance is given by

$$V_t[S_t \mid y_t^i] = \frac{1}{\rho_f + \rho_t^{si}} \qquad i = 1,2 \tag{49}$$

Again traders are myopic, maximizing utility on a period-by-period basis. Demands for each group of traders is given by

$$\rho_f(p_{t-1} - p_t) + \rho^{si}{}_t(y^i{}_t - p_t) \qquad i = 1,2 \tag{50}$$

and the equilibrium price equation can be rewritten as

[7] Some type of rational bubbles may be considered as another possible solution. See, e.g., Hamilton (1985).

$$p_t = \frac{\rho_f\, p_{t-1} + \mu\rho_t^{s1}\overline{y}_t^{\,1} + (1-\mu)\rho_t^{s2}\overline{y}_t^{\,2}}{\rho_f + \mu\rho_t^{s1} + (1-\mu)\rho_t^{s2}} \tag{51}$$

Applying the Strong Law of Large numbers, the average signal will converge to the noisy realization of the fundamental as before,

$$p_t = \frac{\rho_f p_{t-1} + (\mu\rho_t^{s1} + (1-\mu)\rho_t^{s2})\theta_t}{\rho_f + \mu\rho_t^{s1} + (1-\mu)\rho_t^{s2}} \tag{52}$$

This will in turn allow per capita volume to be expressed as in the previous section as

$$\begin{aligned} v_t = {} & \frac{\mu}{2}\left[2\,\frac{\rho^{s1}}{\sqrt{\rho_t^1}}\phi\left(\frac{\hat{\delta}_t^1\sqrt{\rho_t^1}}{\rho_t^{s1}}\right) + \hat{\delta}_t^1\left(\Phi\left(\frac{\hat{\delta}_t^1\sqrt{\rho_t^1}}{\rho_t^{s1}}\right) - \Phi\left(\frac{-\hat{\delta}_t^1\sqrt{\rho_t^1}}{\rho_t^{s1}}\right)\right)\right] \\ & + \frac{(1-\mu)}{2}\left[2\,\frac{\rho^{s2}}{\sqrt{\rho^2}}\phi\left(\frac{\hat{\delta}_t^2\sqrt{\rho_t^2}}{\rho^{s2}}\right) + \hat{\delta}_t^2\left(\Phi\left(\frac{\hat{\delta}_t^2\sqrt{\rho^2}}{\rho^{s2}}\right) - \Phi\left(\frac{-\hat{\delta}_t^2\sqrt{\rho^2}}{\rho^{s2}}\right)\right)\right] \end{aligned} \tag{53}$$

where

$$\hat{\delta}_t^j = \rho_f(p_t - p_{t-1})\left(\frac{\rho_t^{sj}}{\mu\,\rho_t^{s1} + (1-\mu)\rho^{s2}} - 1\right) \tag{54}$$

is the demand of group j, ϕ is the standard normal density, and Φ the cumulative normal distribution function.

Comparative statics can again be performed on any single period, and the results will be comparable to those of the BEO model. For any price, p, per capita volume increases in the precision of group 1's signal for ($\rho^1 < \rho_w$), and decreases in ρ^1 for ($\rho^1 > \rho_w$). The difference here is that ρ_w is slightly more complicated due to the definition of the fundamental. Whereas before θ was defined as

$$\theta_t = \Psi + w_t \tag{55}$$

now S_t replaces Ψ and this expression becomes

$$\theta_t = S_t + w_t \tag{56}$$

where $w_t \sim N(0, 1/\rho_w)$, and $S_t \sim N(S_{t-1}, 1/\rho_f)$. Since $1/\rho_w$ and $1/\rho_f$ are are defined to be uncorrelated, the conditions above require $\rho^1 < (\rho_w + \rho_f)$, for per capita volume to increase in the precision of group 1's signal, and $\rho^1 > (\rho_w + \rho_f)$, for per capita volume to decrease in the precision, ρ^1.

Extending this model to multiple periods demonstrates how the movement of the fundamental each period prevents prices from ever becoming fully revealing. This is because the prior on the previous price cannot be updated each period as in the BEO model. When the true fundamental is stochastic and follows a Markov process, traders only weight the prior with the precision of the most recent period. This was not the situation in the BEO model where the relative weight on the updated prior was increasing each period.

This difference between the BEO model and the stochastic fundamental model may be seen by comparing the variance of the noise term of the stochastic fundamental, ρ_f, with the updating formula for the variance of the prior in the BEO model given as

$$\rho_{t-1} = [\ (t\text{-}1)\ \rho_w + \rho_o] \tag{57}$$

As t increases with each period, the size of the first term in brackets also increases while the second term in brackets remains constant. This causes the entire expression to increase over time. Thus, the precision of the prior is increasing over time.

The result of changing the time series property of the true fundamental is that private signals have equal value over time, and the single period conclusions may be applied to the multiple period model. Traders use Bayesian updating each period and the weight on prior information vs. new information remains constant. Prices cannot become fully revealing because the noisy fundamental always has only one realization for each realization of the true fundamental.

5 Discussion

It has been shown how the original Grossman & Stiglitz (1980) rational expectations model can be modified by altering the underlying price process to study an information structure where traders are asymmetrically informed. Starting with the model of Blume, Easley, and O'Hara (1994), a new model was formulated which incorporated a discrete-time continuous-state fundamental into the model. The time series properties of per capita volume were then be examined in this new formulation. While BEO argue that volume has a non degenerate distribution over time when prices are fully revealing, it has been shown that volume is non degenerate only when the fundamental of the market is stochastic and prices are partially revealing.

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Appendix

Proof of Lemma: In the appendix of BEO, a lemma (Lemma 1) is proposed to be used in the proof of their Proposition 1. Since only a sketch of a proof is provided in BEO, a full derivation is provided here.

Lemma: Let $y \sim N(\theta,\ 1/\rho)$, then

$$E\left[\left|\gamma y + a\right|\right] = +\frac{2\gamma}{\sqrt{2\pi\rho}}\ e^{-\frac{1}{2}\left(\frac{\sqrt{\rho}}{\gamma}\ \delta\right)^2} + \delta\left[\Phi\left(\frac{\delta\sqrt{\rho}}{\gamma}\right) - \Phi\left(\frac{-\delta\sqrt{\rho}}{\gamma}\right)\right]$$

for $\delta = a + \gamma\theta$.

Proof: As noted in BEO,

$$\Pr\left(\left|\gamma y + a\right| \leq \alpha\right) = \Pr\left(-\alpha \leq \gamma y + a \leq \alpha\right)$$

$$= \Pr\left(\left(\frac{\sqrt{\rho}}{\gamma}\left(\alpha - a - \gamma\theta\right)\right) \geq u \geq \left(\frac{\sqrt{\rho}}{\gamma}\left(-\alpha - a - \gamma\theta\right)\right)\right)$$

$$= \Phi\left(\frac{\sqrt{\rho}}{\gamma}\left(\alpha - a - \gamma\theta\right)\right) - \Phi\left(\frac{\sqrt{\rho}}{\gamma}\left(-\alpha - a - \gamma\theta\right)\right),$$

for u, the standard Normal density. This expression can be explained by defining

$$E\left[\gamma y + a\right] = \gamma\theta + a,\ \text{since } E[y] = \theta$$

$$V\left[\gamma y + a\right] = \gamma^2/\rho,\ \text{since } V[y] = 1/\rho$$

then these equations can be transformed into a standard Normal with dummy variable α as

$$\frac{-\alpha - E\left[\gamma y + a\right]}{\sqrt{V\left[\gamma y + a\right]}} \leq u \leq \frac{\alpha - E\left[\gamma y + a\right]}{\sqrt{V\left[\gamma y + a\right]}}$$

and substituting

$$\frac{\sqrt{\rho}}{\gamma}\left(-\alpha - \gamma\theta - a\right) \leq u \leq \frac{\sqrt{\rho}}{\gamma}\left(\alpha - \gamma\theta - a\right).$$

This probability can be thought of as the area under the standard Normal between the specified endpoints. It is convenient to use the cumulative distribution function. First find all the area below the upper endpoint by using the cumulative distribution function, then again using the cumulative distribution function subtract the area below the lower endpoint. The result is the difference of two cumulative distribution functions. Using first the upper endpoint,

$$\Pr[X \le x] = \Pr\left[z \le \frac{x-\mu}{\sigma}\right]$$

or

$$\Phi(u) = \Pr[U \le u]$$

then

$$\Phi\left[\tfrac{\sqrt{\rho}}{\gamma}(\alpha - \gamma\theta - a)\right] = \Pr\left[U \le \tfrac{\sqrt{\rho}}{\gamma}(\alpha - \gamma\theta - a)\right].$$

Similarly, the area below the lower endpoint is

$$\Phi\left[\tfrac{\sqrt{\rho}}{\gamma}(-\alpha - \gamma\theta - a)\right].$$

Subtracting yields

$$\Phi\left[\tfrac{\sqrt{\rho}}{\gamma}(\alpha - \gamma\theta - a)\right] - \Phi\left[\tfrac{\sqrt{\rho}}{\gamma}(-\alpha - \gamma\theta - a)\right].$$

Now the density function can be found by differentiating the cumulative distribution function by use of the chain rule,

$$f(\alpha) = \frac{d}{d\alpha} F(\alpha)$$

$$= \frac{d}{d\alpha}\left\{\tfrac{\sqrt{\rho}}{\gamma}\left[\phi(\alpha - \gamma\theta - a)\right] - \tfrac{\sqrt{\rho}}{\gamma}\left[\phi(-\alpha - \gamma\theta - a)\right]\right\}.$$

Once the density function is known, the expected value of any random variable x can be found by using the definition of expected value,

$$E[x] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

where f(x) is the density function for the random variable x. Since we know the random variable cannot take on negative values due to the absolute value function, this may be simplified to

$$E\left[\alpha\right] = \int_{0}^{\infty} \alpha \, f(\alpha) \, d\alpha$$

where $\alpha = \left|\gamma y + a\right|$, and $f(\alpha)$ is the density function of $\left|\gamma y + a\right|$ from the above calculation. This integral may be solved by a change of variable, but first we define for convenience,

$$b = \frac{\sqrt{\rho}}{\gamma} \quad \text{and} \quad \delta = a + \gamma\theta$$

Then rewrite substituting,

$$F(\alpha) = \Phi_A(\alpha) = \Phi(b(\alpha - \delta)) - \Phi(b(-\alpha - \delta)),$$

and

$$\frac{d\,\Phi_A(\alpha)}{d\,\alpha} = f(\alpha) = b\,\phi(b(\alpha - \delta)) + b\,\phi(b(-\alpha - \delta)).$$

Writing out the density function yields

$$f(\alpha) = \frac{b}{\sqrt{2\pi}} e^{-\frac{1}{2} b^2 (\alpha - \delta)^2} + \frac{b}{\sqrt{2\pi}} e^{-\frac{1}{2} b^2 (-\alpha - \delta)^2}.$$

Now using this density function find the expected value,

$$E\left|\gamma y + a\right| = E\left|\alpha\right| = \int_{0}^{\infty} \alpha \, f(\alpha) \, d\alpha$$

or

$$E\left|\alpha\right| = \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha \, e^{-\frac{1}{2} b^2 (\alpha - \delta)^2} \, d\alpha + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha \, e^{-\frac{1}{2} b^2 (-\alpha - \delta)^2} \, d\alpha .$$

Use a change of variables to simplify the integral

$$z_1 = \alpha - \delta \;\Rightarrow\; \alpha = z_1 + \delta$$

$$z_2 = \alpha + \delta \;\Rightarrow\; \alpha = z_2 - \delta$$

then substituting and separating

$$= \frac{b}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} z_1\, e^{-\frac{1}{2}b^2 z_1^2}\, dz_1 + \delta \int_{-\infty}^{\infty} e^{-\frac{1}{2}b^2 z_1^2}\, dz_1\right] +$$

$$\frac{b}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} z_2\, e^{-\frac{1}{2}b^2 z_2^2}\, dz_2 - \delta \int_{-\infty}^{\infty} e^{-\frac{1}{2}b^2 z_2^2}\, dz_2\right]$$

Rewrite and reorganize as

$$= \frac{b}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} z_1\, e^{-\frac{1}{2}b^2 z_1^2}\, dz_1 + \int_{-\infty}^{\infty} z_2\, e^{-\frac{1}{2}b^2 z_2^2}\, dz_2\right] +$$

$$\frac{b}{\sqrt{2\pi}}\left[\delta \int_{-\infty}^{\infty} e^{-\frac{1}{2}b^2 z_1^2}\, dz_1 - \delta \int_{-\infty}^{\infty} e^{-\frac{1}{2}b^2 z_2^2}\, dz_2\right]$$

The first and second terms will be solved separately. The first term

$$= \frac{b}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} z_1\, e^{-\frac{1}{2}b^2 z_1^2}\, dz_1 + \int_{-\infty}^{\infty} z_2\, e^{-\frac{1}{2}b^2 z_2^2}\, dz_2\right]$$

can be solved by noting that after the substitution

$$z = \sqrt{x} \quad \Rightarrow \quad dz = \frac{1}{2\sqrt{x}}\, dx$$

the following is true

$$\int z\, e^{k z^2} = \int \sqrt{x}\; e^{k x} \frac{1}{2\sqrt{x}}\, dx = \int \frac{1}{2}\, e^{k x} dx\,.$$

Applying this technique yields,

$$\frac{b}{\sqrt{2\pi}}\left[\frac{1}{2\left(-\frac{1}{2}b^2\right)}\, e^{-\frac{1}{2}b^2 z_1^2} + \frac{1}{2\left(-\frac{1}{2}b^2\right)}\, e^{-\frac{1}{2}b^2 z_2^2}\right]\Bigg|_0^{\infty}$$

and replacing original values then simplifying

$$-\frac{1}{b\sqrt{2\pi}}\left[e^{-\frac{1}{2}b^2(\alpha-\delta)^2}+e^{-\frac{1}{2}b^2(\alpha+\delta)^2}\right]\Bigg|_0^{\infty}$$

then

$$+\frac{1}{b\sqrt{2\pi}}\left[e^{-\frac{1}{2}b^2(-\delta)^2}+e^{-\frac{1}{2}b^2(+\delta)^2}\right]$$

and since $(-\delta)^2 = (\delta)^2$,

$$+\frac{2}{b\sqrt{2\pi}}\;e^{-\frac{1}{2}(b\;\delta)^2}$$

replacing substituted values

$$+\frac{2\,\gamma}{\sqrt{2\pi\rho}}\;e^{-\frac{1}{2}\left(\frac{\sqrt{\rho}}{\gamma}\;\delta\right)^2}$$

Now choose the second term from the original expression above

$$=\frac{b}{\sqrt{2\pi}}\left[\delta\int_{-\infty}^{\infty}e^{-\frac{1}{2}b^2z_1^2}\,dz_1-\delta\int_{-\infty}^{\infty}e^{-\frac{1}{2}b^2z_2^2}\,dz_2\right]$$

Rewrite substituting back out z,

$$\frac{\delta\,b}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}e^{-\frac{1}{2}b^2(\alpha-\delta)^2}\,d\alpha-\int_{-\infty}^{\infty}e^{-\frac{1}{2}b^2(\alpha+\delta)^2}\,d\alpha\right]$$

and change limits of integration to reflect the absolute value function in the original equation. Also change the order of integration and sign of each term

$$\frac{\delta\,b}{\sqrt{2\pi}}\left[-\int_{-\infty}^{0}e^{-\frac{1}{2}b^2(\alpha-\delta)^2}\,d\alpha+\int_{-\infty}^{0}e^{-\frac{1}{2}b^2(\alpha+\delta)^2}\,d\alpha\right]$$

substitute back out b,

$$\frac{\delta\sqrt{\rho}}{\gamma\sqrt{2\pi}}\left[-\int_{-\infty}^{0} e^{-\frac{1}{2}\left[\frac{\sqrt{\rho}\,(\alpha-\delta)}{\gamma}\right]^{2}} d\alpha + \int_{-\infty}^{0} e^{-\frac{1}{2}\left[\frac{\sqrt{\rho}\,(\alpha+\delta)}{\gamma}\right]^{2}} d\alpha\right]$$

and define

$$w_1 = \frac{\sqrt{\rho}\,(\alpha-\delta)}{\gamma} \quad \text{and} \quad w_2 = \frac{\sqrt{\rho}\,(\alpha+\delta)}{\gamma}$$

then

$$d\,w = \frac{\sqrt{\rho}}{\gamma}\,d\,\alpha \quad \Rightarrow \quad d\,\alpha = \frac{\gamma}{\sqrt{\rho}}\,d\,w$$

implies

$$\alpha = 0 \quad \Rightarrow \quad w_1 = \frac{-\delta\sqrt{\rho}}{\gamma} \quad \text{and} \quad w_2 = \frac{\delta\sqrt{\rho}}{\gamma}$$

$$\alpha = -\infty \quad \Rightarrow \quad w_1 = -\infty \quad \text{and} \quad w_2 = -\infty$$

Using these

$$\frac{\delta\sqrt{\rho}}{\gamma\sqrt{2\pi}}\left[-\int_{-\infty}^{\frac{-\delta\sqrt{\rho}}{\gamma}} e^{-\frac{1}{2}(w_1)^2}\frac{\gamma}{\delta\sqrt{\rho}}\,dw_1 + \int_{-\infty}^{\frac{\delta\sqrt{\rho}}{\gamma}} e^{-\frac{1}{2}b^2(w_2)^2}\frac{\gamma}{\delta\sqrt{\rho}}\,dw_2\right]$$

simplifying

$$\delta\left[-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\delta\sqrt{\rho}}{\gamma}} e^{-\frac{1}{2}(w_1)^2}\,dw_1 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\delta\sqrt{\rho}}{\gamma}} e^{-\frac{1}{2}b^2(w_2)^2}\,dw_2\right]$$

And using the definition of the cumulative distribution function

$$\delta\left[-\Phi\left(\frac{-\delta\sqrt{\rho}}{\gamma}\right) + \Phi\left(\frac{\delta\sqrt{\rho}}{\gamma}\right)\right]$$

Recombining the two pairs of integrals

$$\frac{2\gamma}{\sqrt{2\pi\rho}}\,e^{-\frac{1}{2}\left(\frac{\sqrt{\rho}}{\gamma}\delta\right)^2} + \delta\left[\Phi\left(\frac{\delta\sqrt{\rho}}{\gamma}\right) - \Phi\left(\frac{-\delta\sqrt{\rho}}{\gamma}\right)\right] \qquad \text{Ä}$$

Derivation of the Equilibrium Price Equation: The derivation of the equilibrium price equation begins with individual trader demand for the risky asset. Individual demands are arrived at by maximizing a negative exponential utility function. Individual demands are then summed over all traders, and a market clearing condition is imposed.

The individual demand for the risky asset is

$$d^i = \frac{\mathrm{E}[\psi \mid H^i] - p}{\mathrm{Var}[[\psi \mid H^i]]}$$

where d^i is the individual demand for trader i, and the conditional expectation and variance of the value of the risky asset given the available information are

$$\mathrm{E}[\psi \mid H^i] \text{ and } \mathrm{Var}[[\psi \mid H^i]].$$

When a prior signal is available, demand for each type of trader takes into account the prior and current signal through Bayesian updating. The demand by the informed and uninformed trader in period 1 is

$$\rho_o [\psi_o - p_1] + \rho_1^{s1} [y_1^i - p_1]$$

and

$$\rho_o [\psi_o - p_1] + \rho_1^{s2} [y_1^i - p_1]$$

where the precision (inverse of variance) for the prior, the informed trader, and the uninformed trader are denoted by ρ_o, ρ_1^{s1}, and ρ_1^{s2}, and the prior and current signals are ψ_o and y_1^i.

Summing the individual demands over both kinds of traders yields

$$\sum_{i=1}^{N} d^i = \sum_{i=1}^{N} \rho_o [\psi_o - p_1] + \mu \sum_{i=1}^{N} \rho_1^{s1} [y_1^i - p_1] + [1 - \mu] \sum_{i=1}^{N} \rho_1^{s2} [y_1^i - p_1].$$

The market clearing condition that excess demand must be zero can now be applied. Also, divide both sides by N

$$0 = \mathrm{N}\,\frac{\rho_o [\psi_o - p_1]}{\mathrm{N}} + \frac{\mu \rho_1^{s1} \left[\sum_{i=1}^{N} y_1^i - p_1 \right]}{\mathrm{N}} + \frac{[1 - \mu] \rho_1^{s2} \left[\sum_{i=1}^{N} y_1^i - p_1 \right]}{\mathrm{N}}.$$

Then define

$$\frac{\sum_{i=1}^{N} y^i}{N} = \bar{y}$$

and substitute into the above to arrive at

$$0 = \rho_o(\psi_o - p_1) + \mu\rho_1^{s1}(\bar{y}_1^1 - p_1) + (1-\mu)\rho_1^{s2}(\bar{y}_1^2 - p_1) .$$

The first period price can now be solved for as

$$p_1 = \frac{\rho_o\psi_o + \mu\rho_1^{s1}\bar{y}_1^1 + (1-\mu)\rho_1^{s2}\bar{y}_1^2}{\rho_o + \mu\rho_1^{s1} + (1-\mu)\rho_1^{s2}} ,$$

and by applying the Law of Large Numbers,

$$\lim_{N\to\infty} \bar{y}^i = \theta$$

the equilibrium price is

$$p_1 = \frac{\rho_o\psi_o + (\mu\rho_1^{s1} + (1-\mu)\rho_1^{s2})\theta}{\rho_o + \mu\rho_1^{s1} + (1-\mu)\rho_1^{s2}} . \qquad \square$$

Figure I

Matrix of Information Structures

Used in Rational Expectation Trading Models

	Uniform	Dispersed
Symmetric		Diamond & Verrecchia Brown & Jennings
Asymmetric	Grossman & Stiglitz Romer I	Blume, Easley & O'Hara Romer II

Figure II

Precision - Volume Relation