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James A. Ohlson

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RISK-AVERSION AND THE MARTINGALE PROPERTY OF STOCK PRICES: COMMENTS*

BY JAMES A. OHLSON[1]

1. INTRODUCTION

An interesting and thought-provoking paper by LeRoy [2] considers the problem of deriving an endogenous intertemporal distribution of risky assets. Specifically, the important question arises whether, or under what assumptions, the induced distribution of security prices will satisfy the martingale property. LeRoy develops a model in which there are two assets, one riskfree and one risky, and then proceeds deriving an endogenous probability distribution of the risky asset. The additional assumptions employed are those of completely homogeneous investors with (identical) constant absolute risk-aversion; further, the equilibrium model is "driven" by an exogenous and stochastic dividend process. (Dividends are assumed to be identical to earnings.) This dividend process follows a linear first-order autoregressive specification. The analysis of the solution to the model developed then shows that the martingale property will not obtain.

The above conclusion is undoubtedly correct for the specific set of assumptions — regarding investors' preferences and the dividend process — used in LeRoy's formal analysis. However, it will be shown here that a simple modification of these assumptions will in fact yield the martingale property; i.e., the return distribution of the risky asset will remain constant for all states of the system. Furtheremore, this result will be derived using more realistic assumptions. It will be postulated that investors have constant relative risk-aversion rather than constant absolute risk-aversion; the percentage change in dividends distributed are assumed to have constant mean and variance rather than assuming that dividends are generated by a first-order linear autoregressive process.

2. LEROY'S MODEL

The basic economic setting is one in which there are n homogeneous investors: they all have identical endowments, preferences, and beliefs. The risk preferences are represented by means and variances of end-of-next-period wealth. The objects of choice are: (i) a risky security, i.e., a "market portfolio," and (ii) a risk-free asset, "cash." The return earned on the risk-free asset is exogenous in the model and the price on the risky security is endogenous. The information set

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used by investors in formulating their *ex ante* probability beliefs regarding next period price and dividends is current and past dividends.

The following notation is used here (since all investors are homogeneous there will be no need to sub- or superscript for the i-th investor):

c = holdings of the risk-free asset ("cash").
r^* = the exogenous risk-free rate; an intertemporal constant.
p_t = price (per share) of the risky asset at time t.
w_t = the i-th investor's wealth at time t.
a_t = proportion of wealth invested in the risky asset at t.
x_t = dividends (per share) paid at t.
$f(.)$ = the endogenous price — valuation function of the risky asset as a function of current and past dividends: $p_t = f(x_t, x_{t-1}, \ldots)$.
$E[\tilde{w}_{t+1}/x_t, \ldots] = E[w_t(a_t(\tilde{p}_{t+1} + \tilde{x}_{t+1})p_t^{-1} + (1 - a_t)(1 + r^*))/.]$ = expected end-of-period wealth $(t+1)$ conditioned on the available information (state of the system) at time t.
$V[\tilde{w}_{t+1}/.]$ = conditional variance of end-of-period wealth.
$U(E[\tilde{w}_{t+1}/.], V[\tilde{w}_{t+1}/.])$ = utility of end-of-period wealth.

For simplicity, but without any loss of generality, assume that for all t there are n investors and n shares outstanding. (LeRoy assumes there is one share outstanding of the risky stock; clearly, this will make no substantive difference in subsequent analysis.) At time t the typical investor's optimal portfolio, and dollar demand for the risky asset, is obtained by maximizing U with respect to $w_t a_t$. As a first-order condition one then gets:[2]

$$(1)\quad U_1(.,.)\{E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/.]p_t^{-1} - (1 + r^*)\} + 2U_2(.,.)w_t a_t V[\tilde{p}_{t+1} + \tilde{x}_{t+1}/.]p_t^{-2} = 0.$$

In equilibrium it is required that all markets clear; in the homogeneous world assumed this implies that each investor holds one share of the risky asset. It follows immediately that $a_t = p_t/w_t$ in equilibrium. Substituting $w_t a_t = p_t$ into (1) and simplifying:

$$(2)\quad U_1(.,.)\{E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/.] - (1 + r^*)p_t\} - 2U_2(.,.)V[\tilde{p}_{t+1} + \tilde{x}_{t+1}/.] = 0.$$

The above market equilibrium condition is probabilistic in the sense that it must hold for all states, "dots," that may occur. (In this case for any realization of the random sequence$\ldots\tilde{x}_{t-2}, \tilde{x}_{t-1}, \tilde{x}_t$.) This is a crucial fact; as LeRoy observes, this implies that the solution to the valuation function, $p_t = f(x_t, \ldots)$, is directly related to the market clearing requirement, (2), and the stochastic process that generates the dividends. In order to develop some explicit closed-form results to f, LeRoy makes two additional assumptions:

[2] LeRoy actually maximizes over a variable $h_t^i \equiv$ the fraction of total risk-stock owned by the i-th investor. To some extent this obscures the basic equilibrium condition when one wishes to consider the case of constant relative risk-aversion. See Section 3 below.

ASSUMPTION 1. U_1/U_2 is a constant.

ASSUMPTION 2. $\tilde{x}_t = \lambda x_{t-1} + \mu + \varepsilon_t$, where $|\lambda| < 1$, $E\varepsilon_t = 0$, $\mu > 0$, $E\varepsilon_t^2 = \sigma^2$; the ε_t's are serially independent.

As a trial solution to the valuation function LeRoy considers $p_t = \alpha + \beta x_t$. Thus,

$$E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] = \alpha + (\beta + 1)E[\tilde{x}_{t+1}/\cdot] = \alpha + (\beta + 1)(\mu + \lambda x_t),$$

$$V[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] = (\beta + 1)^2\sigma^2.$$

Substituting these two expressions and $p_t = \alpha + \beta x_t$ into the equilibrium condition (2) yields

$$\alpha + (\beta + 1)(\mu + \lambda x_t) - (1 + r^*)(\alpha + \beta x_t) - (\beta + 1)^2\sigma^2 2U_2/U_1 = 0,$$

which must hold for all x_t that may occur.

It is, therefore, required that

$$(\beta + 1)\lambda - (1 + r^*)\beta = 0,$$

$$(\beta + 1)\mu - \alpha r^* - (\beta + 1)^2\sigma^2 2U_2/U_1 = 0.$$

Solving explicitly for α and β (as a function of λ, μ, σ^2, and U_2/U_1), it is readily shown that, in general

$$E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] \neq \text{constant } p_t,$$

where the constant should be independent of the state of the system. Therefore, as LeRoy indeed concludes, with this particular set of assumptions, there is no reason to believe that the endogenous price-distribution will satisfy the martingale property. Specifically, the martingale property holds if and only if investors are risk-neutral.[3]

3. AN ALTERNATIVE SET OF SPECIFICATIONS

The assumption that investors exhibit constant absolute risk-aversion, as well as the particular dividend process considered, is, of course, not realistic. From the theoretical point of view, however, there is a more important issue relating to the character of the expected utility function: if the martingale property does *not* obtain, then it is undesirable to assume that the one-period expected utility function does not depend on the state of the system. In Fama [**1**] and Merton [**3**] it is demonstrated that a critical necessary assumption such that the investor can be presumed to have a one-period (derived or induced) expected utility function in two parameters (mean and variance) is that the investment opportunity set remains constant over time; i.e., with one risky asset, r^*, $E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot]/p_t$ and $V[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot]/p_t^2$ must be constants independent of t and the '.'. In other words, it is unsatisfactory to derive an endogenous intertemporal dis-

[3] See [3, (444)].

tribution of returns which does not reflect demands for the risky asset in conjunction with the probabilistic shifts in the investment opportunity set. It also follows that it is undesirable to use one-period utility functions which do not reflect such changing investment environments, unless the endogenous solution in fact shows that the investment opportunity set will remain constant (with probability one). To develop a set of assumptions such that the exact martingale property obtains is thus of more than a cursory interest. Below it is shown that a more realistic set of assumptions will indeed imply that the martingale property is satisfied. Consider:

ASSUMPTION 1′. $-U_1/2U_2w_t = \text{constant} \equiv k^{-1} > 0$; all w_t, $E[\tilde{w}_{t+1}/\cdot]$, and $V[\tilde{w}_{t+1}/\cdot]$.

ASSUMPTION 2′. $(\tilde{x}_{t+1} - x_t)/x_t = \theta + \varepsilon_{t+1}$; $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2$; the ε_t's are serially independent.

ASSUMPTION 3′. $c = 0$; i.e., r^* reflects the (constant) private borrowing-lending rate; x_t is consumed in the period that begins at t.

Comparing the above assumptions with those of LeRoy, we first note that Assumption 1′ corresponds to constant relative risk-aversion just as Assumption 1 is related to constant absolute risk-aversion. Specifically, consider $Eu(\tilde{w}_{t+1})$ where u is the usual cardinal utility of wealth function and define U as the approximation of $Eu(\tilde{w}_{t+1})$ after a second-order Taylor's expansion:[4]

$$U \equiv u(w_t) + u'(w_t)E[\tilde{w}_{t+1} - w_t] - 1/2u''(w_t)V[\tilde{w}_{t+1}].$$

Assumption 1 is now equivalent to $u'(Y_t)/u''(Y_t) = \text{constant}$, i.e., $u(Y) = -\exp\{-\alpha Y\}$, $\alpha > 0$; Assumption 1′ is equivalent to $Yu''(Y)/u'(Y) = \text{constant}$, i.e., $u(Y) = Y^\alpha/\alpha$, $\alpha < 1$, or $u(Y) = \log Y$. Assumption 2′ simply asserts that the percentage changes in dividends have purely random increments with mean θ. A third assumption, Assumption 3′, now also becomes crucial. In the LeRoy analysis no similar assumption is required because of Assumption 1; if an investor has constant absolute risk-aversion, then the optimal dollar amount invested in the risky asset is independent of the total amount to be invested in the two assets. That is, $a_t(w_t)w_t = \text{constant}$ for all w_t, p_t fixed. This is immediate from (1) and a well-known property of constant absolute risk-aversion. The equilibrium price, p_t, is therefore independent of aggregate wealth; changes in the endowments of cash and/or current and past dividends do not affect the equilibrium conditions. Of course, this will further require that the "government" satisfies the demand and supply for risk-free funds if a fixed and constant interest rate of r^* is to be sustained.

[4] The justification for $E[\tilde{w}_{t+1} - w_t]^2 \doteq V[\tilde{w}_{t+1}]$ is implied by the fundamental mean-variance approximation theorem as set forth by Samuelson [4]. In any case, the only point which is really important here is that one can make "plausible" assumptions such that $w_t U_2/U_1 = \text{constant}$.

Assumption 1′ implies that the optimal proportion invested in the risky asset is independent in (aggregate) wealth. Hence, Assumption 3′ now entails that in equilibrium no lending — or borrowing — occurs since all investors are homogeneous. It is immediate that, in equilibrium, with n investors and n shares outstanding, one has $p_t = w_t$, or, simply, $a = 1$. Conversely, if $c > 0$, then the actual supply of c becomes a critical ingredient in the analysis and explicit solutions will almost certainly be difficult to obtain unless one can assume directly that a remains fixed from period to period. With $c = 0$ a plausible rationale for this is provided if it assumed that x_t is allocated to consumption.

With fixed r^* a solution to the valuation function can now be derived if moderate restrictions are imposed on θ, σ^2, r^*, and k; the solution will have the characteristics that $E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] = \text{const}\, p_t$ for every state of the system. As a trial solution to $p_t = f(x_t)$ consider $p_t = \alpha x_t$. Then $E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] = (\alpha+1)E[\tilde{x}_{t+1}/\cdot] = (\alpha+1)(\theta+1)x_t$ and $V[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] = (\alpha+1)^2\sigma^2 x_t^2$. Using Assumptions 1′–3′ one thus obtains as an equilibrium condition (1):[5]

$$(\alpha + 1)(\theta + 1)\alpha^{-1} - (1 + r^*) = k(\alpha + 1)^2\sigma^2\alpha^{-2}.$$

The above is a second-order polynomial in α and it is straightforward to verify that a real solution $p_t = \alpha(\theta, \sigma^2, r^*, k)x_t$ exists given mild restrictions on the exogenous parameters. Moreover, provided a real solution exists,

$$E[\tilde{p}_{t+1} + \tilde{x}_{t+1}/\cdot] = (\alpha + 1)(\theta + 1)x_t = (\alpha + 1)(\theta + 1)^{-1}\alpha^{-1}p_t$$

for every realization of $\cdots\tilde{x}_{t-2}, \tilde{x}_{t-1}, \tilde{x}_t$.

In the development given here, as well as in that of LeRoy, it has been assumed that r^* is fixed. In an extended equilibrium analysis this is untenable, and by the introduction of consumption and utility of consumption it is possible to consider r^* as endogenous. However, this would appear to complicate matters considerably in terms of developing explicit solutions. In addition, if r^* does indeed change over time then there is little reason to believe that the strict martingale can ever hold in a theoretical analysis.

The model developed above clearly uses very special assumptions. It is therefore well to note that it does not impair LeRoy's major contention that the martingale hypothesis can be obtained only as a special case. The point here has merely been to show that the martingale property is also easily derived without employing assumptions which are comparatively restrictive. From a theoretical point of view both sets of assumptions are restrictive; from an empirical point of view the assumptions used here would seem more plausible as a first order approximation.

University of California, Berkeley, U.S.A.

[5] Alternatively, constant absolute risk-aversion implies that, in equilibrium, E [Rate of return/.] $- r^* = \text{const}\; p_t V$ [Rate of return/.]. Constant relative risk-aversion implies that 'p_t' disappears in the expression and E [Rate of return/.] $- r^* = \text{const}\; V$ [Rate of return/.].

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