On the Efficiency of Competitive Stock Markets Where Trades Have Diverse Information

Sanford Grossman

ON THE EFFICIENCY OF COMPETITIVE STOCK MARKETS WHERE TRADES HAVE DIVERSE INFORMATION

SANFORD GROSSMAN*

1. Introduction

I have shown elsewhere that competitive markets can be “over-informationally” efficient. (See Grossman [1975] for this and a review of other work in this area.) If competitive prices reveal too much information, traders may not be able to earn a return on their investment in information. This was demonstrated for a market with two types of traders, “informed” and “uninformed.” “Informed” traders learn the true underlying probability distribution which generates a future price, and they take a position in the market based on this information. When all informed traders do this, current prices are affected. “Uninformed” traders invest no resources in collecting information, but they know that current prices reflect the information of informed traders. Uninformed traders form their beliefs about a future price from the information of informed traders which they learn from observing current prices.

In the above framework, prices transmit information. However, it is often claimed that prices aggregate information. In this paper we analyze a market where there are n-types of informed traders. Each gets a “piece of information.” In a simple model we study the operation of the price system as an aggregator of the different pieces of information.

We consider a market where there are two assets; a risk-free asset and a risky asset. Each unit of the risky asset yields a return of \( \tilde{P}_1 \) dollars. \( \tilde{P}_1 \) will also be referred to as the price of the risky asset in period 1. In period 0 (the current period), each trader gets information about \( \tilde{P}_1 \) and then decides how much of risky and non-risky assets to hold. This determines a current price of the risky asset, \( P_0 \), which will depend on the information received by all traders. We assume that the \( i \)th trader observes \( y_i \), where \( y_i = P_1 + \epsilon_i \). There is a noise term, \( \epsilon_i \), which prevents any trader from learning the true value of \( P_1 \). The current equilibrium price is a function of \((y_1, y_2, \ldots, y_n)\); write it as \( P_0(y_1, y_2, \ldots, y_n) \).

The main result of this paper is that when there are n-types of traders \((n > 1)\), \( P_0 \) reveals information to each trader which is of “higher quality” than his own information. That is, the competitive system aggregates all the market’s information in such a way that the equilibrium price summarizes all the information in the

* Department of Economics and Graduate School of Business, Stanford University. I am grateful to Michael Rothschild, Joseph Stiglitz and the participants of the Summer Seminar 1975 at the Institute for Mathematical Studies in the Social Sciences, Stanford University for their helpful comments. This work was supported by National Science Foundation Grant SOC74-11446 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, and the Dean Witter Foundation. Due to space limitations, an Appendix on the subject of the “Uniqueness of Equilibrium” is not included in the article and is available from the author upon request.
market. $P_0(y_1, y_2, \ldots, y_n)$ is a sufficient statistic for the unknown value of $P_1$. A trader who invests nothing in information and observes the market price can achieve a utility as high as traders who pay for the information $y_i$. Similarly, a trader who purchases $y_i$ and then observes $P_0(y)$ (where $y \equiv (y_1, y_2, \ldots, y_n)$), finds that $y_i$ is redundant; $P_0(y)$ contains all the information he requires. That is, informationally efficient price systems aggregate diverse information perfectly, but in doing this the price system eliminates the private incentive for collecting the information.

The above result is demonstrated in the context of a simple mean-variance model. The result that the price system perfectly aggregates information is not robust. This is shown in the context of the above model when "noise" is added. One example of "noise" is an uncertain total stock of the risky asset. However, the paradoxical nature of "perfect markets," which the model illustrates, is robust. When a price system is a perfect aggregator of information it removes private incentives to collect information. If information is costly, there must be noise in the price system so that traders can earn a return on information gathering. If there is no noise and information collection is costly, then a perfect competitive market will break down because no equilibrium exists where information collectors earn a return on their information, and no equilibrium exists where no one collects information. The latter part follows from the fact that if no one collects information then there is an incentive for a given individual to collect costly information because he does not affect the equilibrium price. When many individuals attempt to earn a return on information collection, the equilibrium price is affected and it perfectly aggregates their information. This provides an incentive for individuals to stop collecting information. In Grossman [1975] there is a more detailed analysis of the breakdown of markets when price systems reveal too much information.

On the other hand, when there is noise so that the price system does not aggregate information perfectly, the allocative efficiency properties of a competitive equilibrium may break down. Hayek [1945] argues that the essence of a competitive price system is that when a commodity becomes scarce its price rises and this induces people to consume less of the commodity and to invest more in the production of the commodity. Individuals need not know why the price has risen, the fact that there is a higher price induces them to counteract the scarcity in an efficient way. This argument breaks down when the price system is noisy. We will show that in such cases each individual wants to know why the price has risen (i.e., what exogenous factors make the price unusually high), and that an optimal allocation of resources involves knowing why the price has risen (i.e., knowledge of the states of nature determining current prices is required).

2. The Model

Assume that trader "i" has an initial wealth $W_0$. Using $W_0$, he can purchase two assets; a risk free asset and a risky asset. His wealth in period 1, $W_1$, is given by

$$\tilde{W}_1 = (1 + r)X_{F1} + \tilde{P}_1 X_i,$$  (1)

where $X_{F1}$ is the value of risk free assets purchased in period 0, $X_i$ is the number of
units of risky assets purchased in period 0, \( r > 0 \) is the exogenous rate of return on the risk free asset, and \( \tilde{P}_1 \) is the (unknown) exogenous payoff per unit on the risky asset (also called the period 1 price of the risky asset). The budget constraint is

\[
W_{0i} = X_{F_i} + P_0 X_i,
\]

where \( P_0 \) is the current price of the risky asset. Substituting (2) into (1) to eliminate \( X_{F_i} \) yields:

\[
\tilde{W}_{1i} = (1 + r) W_{0i} + \left[ \tilde{P}_1 - (1 + r) P_0 \right] X_i.
\]

At time zero, \( P_1 \) is unknown. The \( i \)th trader observes \( y_i \), where

\[
y_i = P_1 + \epsilon_i,
\]

and \( P_1 \) is a realization of the random variable \( \tilde{P}_1 \). Thus, a fixed, but unknown, realization of \( \tilde{P}_1 \) mixes with noise, \( \epsilon_i \), to produce the observed \( y_i \). Later, we shall argue that traders also get information from \( P_0 \). For the present, let \( I_i \) denote the information available to the \( i \)th trader. Assume that the \( i \)th trader has a utility function

\[
U_i(\tilde{W}_{1i}) = -e^{-a_i \tilde{W}_{1i}}, \quad a_i > 0,
\]

where \( a_i \) is the coefficient of absolute risk aversion. Each trader is assumed to maximize the expected value of \( U_i(\tilde{W}_{1i}) \) conditional on \( I_i \). If \( \tilde{W}_{1i} \) is normally distributed conditional on \( I_i \), then

\[
E \left[ U_i(\tilde{W}_{1i}) \middle| I_i \right] = -\exp \left\{ -a_i \left[ E \left[ \tilde{W}_{1i} \middle| I_i \right] - \frac{a_i}{2} \text{Var} \left[ \tilde{W}_{1i} \middle| I_i \right] \right] \right\},
\]

where \( \text{Var} \left[ \tilde{W}_{1i} \middle| I_i \right] \) is the conditional variance of \( \tilde{W}_{1i} \) given \( I_i \). It follows that to maximize \( E \left[ U_i(\tilde{W}_{1i}) \middle| I_i \right] \) is equivalent to maximizing

\[
E \left[ \tilde{W}_{1i} \middle| I_i \right] - \frac{a_i}{2} \text{Var} \left[ \tilde{W}_{1i} \middle| I_i \right],
\]

since the expression in (7) in a monotone increasing transformation of the expression in (6). All we have shown is that mean-variance analysis in the Normal case can be derived from the utility function in (5).

From (3),

\[
E \left[ \tilde{W}_{1i} \middle| I_i \right] = (1 + r) W_{0i} + \left\{ E \left[ \tilde{P}_1 \middle| I_i \right] - (1 + r) P_0 \right\} X_i
\]
and

$$\text{Var} \left[ \tilde{W}_1 | I_i \right] = \bar{X}_i^2 \text{Var} \left[ \tilde{P}_1 | I_i \right]. \tag{9}$$

In deriving (8) and (9) we have used the fact that $W_{0i}$, $r$, and $P_0$ are known to the firm in period 0. Thus, from (7)–(9), the consumer’s problem is to maximize

$$(1 + r)W_{0i} + \left( E \left[ \tilde{P}_1 | I_i \right] - (1 + r)P_0 \right) X_i - \frac{a_i}{2} X_i^2 \text{Var} \left[ \tilde{P}_1 | I_i \right] \tag{10}$$

by choosing $X_i$. Using the calculus, an optimal $X_i$, $X_i^d$, satisfies

$$X_i^d = \frac{E \left[ \tilde{P}_1 | I_i \right] - (1 + r)P_0}{a_i \text{Var} \left[ \tilde{P}_1 | I_i \right]} \tag{11}.$$

Thus, the demand for the risky asset depends on its expected price appreciation and on its variance. Let $\bar{X}$ be the total stock of the risky asset. An equilibrium price in period 0 must cause $\sum_{i=1}^{n} X_i^d = \bar{X}$. From (11), the $i$th trader’s demand for the risky asset depends on the information he receives. This depends on the observation he gets, $y_i$. Thus, since the total demand for the risky asset depends on $y_1, y_2, \ldots, y_n$, it is natural to think of the market clearing price as depending on the $y_i$, $i = 1, 2, \ldots, n$. Let $y = (y_1, y_2, \ldots, y_n)$, then the equilibrium price is some function of $y$, $P_0(y)$. That is, different information about the return on an asset leads to a different equilibrium price of the asset.

There are many different functions of $y$. For a particular function, $P_0^*(y)$ to be an equilibrium we require that: for all $y$,

$$\sum_{i=1}^{n} \left( \frac{E \left[ \tilde{P}_1 | y_i, P_0^*(y) \right] - (1 + r)P_0^*(y)}{a_i \text{Var} \left[ \tilde{P}_1 | y_i, P_0^*(y) \right]} \right) = \bar{X}. \tag{12}$$

(12) states that the total demand for the risky asset must equal the total supply for each $y$. (Throughout we put no non-negativity constraint on prices. By proper choice of parameters the probability of a negative price can be made arbitrarily small.) The $i$th trader’s demand function under the price system $P_0^*(y)$ is

$$X_i^d[P_0^*(y)] = \frac{E \left[ \tilde{P}_1 | y_i, P_0^*(y) \right] - (1 + r)P_0^*}{a_i \text{Var} \left[ \tilde{P}_1 | y_i, P_0^*(y) \right]} \tag{13}.$$

The $i$th trader’s information $I_i$, is $y_i$ and $P_0^*(y_i)$. He is able to observe his own sample $y_i$ and $P_0^*(y_i)$. $P_0^*(y_i)$ gives the $i$th trader some information about the sample
of other traders. The next section shows that \( P_0^*(y) \) reveals "all" the information of the traders.

\( P_0^*(y) \) can be interpreted as a stationary point of the following process. Suppose traders initially begin in a naive way, thinking of \( P_0 \) as a number and conditioning only on \( y_i \). Let an auctioneer call out prices until the market clears. Call this solution \( \bar{P}_0(y) \). That is \( \bar{P}_0(y) \) solves

\[
\sum_{i=1}^n \frac{E[\bar{P}_1|y_i] - (1 + r)\bar{P}_0(y)}{a_i \text{Var}[\bar{P}_1|y_i]} = \bar{X}. \tag{13a}
\]

Each period traders come to the market with another realization of \( \bar{y} \), and another \( \bar{P}_0(y) \) is found where the auction stops. After many repetitions traders can tabulate the empirical distribution of \((\bar{P}_0, \bar{P}_1)\) pairs. From this they get a good estimate of the joint distribution of \( \bar{P}_0 \) and \( \bar{P}_1 \). After this joint distribution is learned, traders will have an incentive to change their bids just as the market is about to clear. This follows from the fact that if everyone observes that the market is about to clear at \( \bar{P}_0(y) \), they can condition their beliefs on \( \bar{P}_0(y) \) and learn something more about \( P_1 \). This changes their demands and thus the market will not clear at \( \bar{P}_0(y) \). Suppose instead that the market has been clearing for a long time with prices generated by \( P_0^*(y) \), a solution to (12). Then at any particular time, given that traders come to the market with some \( y_i \), if the market is about to clear at \( P_0^*(y) \), and traders then realize that \( P_0^*(y) \) is the equilibrium, they will not change their bids due to the new information they get about \( P_1 \) from \( P_0^*(y) \). \( P_0^*(y) \) is a self fulfilling expectations equilibrium: when all traders think prices are generated by \( P_0^*(y) \), they will act in such a way that the market clears at \( P_0^*(y) \).

3. There is an Equilibrium Price Which is a Sufficient Statistic

Assume that in (4), \( \epsilon_i \) is a random variable which is normally distributed, with mean 0 and variance 1. Thus, each trader "i" observes \( y_i = P_1 + \epsilon_i \), and given \( P_1 \), \( y_i \) is Normal with mean \( P_1 \) and variance 1. Each trader gets information of equal precision in that \( \text{Var}\epsilon = 1 \) for each trader "i". Further assume that \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) is jointly normally distributed and covariance \((\epsilon_i, \epsilon_j) = 0 \) if \( i \neq j \). Thus, we assume that the joint density of \( y \) given \( P_1 \), say \( f(y|P_1) \), is multivariate Normal with mean vector \((P_1, P_1, P_1, \ldots, P_1)\) and covariance matrix which is the identity matrix. \( P_1 \) is assumed unknown at time zero, however, traders believe that \( P_1 \) is distributed independently of \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) and \( P_1 \) is Normal \((P_1, \sigma^2)\). This marginal distribution of \( P_1 \) has two interpretations. Under a Bayesian interpretation, next period's price is some fixed number, and people represent their uncertainty about the value of that number with a prior distribution which is Normal \((P_1, \sigma^2)\). A non-Bayesian interpretation is that nature draws the true price next period from an urn with distribution Normal \((P_1, \sigma^2)\). Nature makes the drawing before period 0. After a
particular $P_i$ is drawn, traders do their research and the $i$th type trader is able to learn the true value of $P_i$ to within $\varepsilon_i$, where $\varepsilon_i$ is distributed as Normal $\sim N(0, 1)$. Under either interpretation, the following is true:

**Theorem 1.** Under the above assumption about the joint distribution of $y$ and $P_1$, if $P_0^*(y)$ is given by

$$P_0^*(y) = \alpha_0 + \alpha_1 \bar{y}, \quad \text{where}$$

$$\bar{y} \equiv \frac{\sum_{i=1}^{n} y_i}{n}, \quad \text{and}$$

$$\bar{P}_1 \equiv \sum_{i=1}^{n} \frac{1}{a_i} - \sigma^2 \bar{X}$$

$$\alpha_0 = \frac{\bar{P}_1 \sum_{i=1}^{n} \frac{1}{a_i} - \sigma^2 \bar{X}}{(1 + n\sigma^2)(1 + r) \sum_{i=1}^{n} \frac{1}{a_i}}, \quad \text{and}$$

$$\alpha_1 = \frac{n\sigma^2}{(1 + n\sigma^2)(1 + r)}.$$  

then $P_0^*(y)$ is an equilibrium. That is, it is a solution to (12).

Before proving the theorem we present some comments on its significance. First $\bar{y}$ is the sample mean of the $y_i$. The equilibrium price depends on the information $y$ only through $\bar{y}$. Second, any trader by observing the value of $P_0^*(y)$ can learn $\bar{y}$ from (14), since by (17), $\alpha_1 > 0$. $\bar{y}$ is a more precise estimate of $P_1$, than $y_i$. Thus the market price aggregates all the information collected by the traders in an “optimal” way. $\bar{y}$ is a sufficient statistic for the family of densities $f(y|P_1)$. The market aggregation is optimal to the extent that it produces a sufficient statistic.

The following lemma is used to prove the theorem:

**Lemma 1.** If $h_i(y_i, \bar{y} | P_1)$ is the joint density of $\bar{y}$ and $y_i$ conditional on $P_1$, then there are functions $g_1(\cdot)$ and $g_2(\cdot)$ such that, for all $y_i$ and $\bar{y} = \sum_{i=1}^{n} y_i/n$,

$$h_i(y_i, \bar{y} | P_1) = g_1(y_i, \bar{y}) g_2(\bar{y}, P_1).$$

That is, $\bar{y}$ is a sufficient statistic for $h_i(y_i, \bar{y} | P_1)$.

**Proof.** Conditional on $P_1$, $y_i$ is Normal $\sim N(\bar{P}_1, 1)$ and $\bar{y}$ is Normal $\sim N(P_1, 1/n)$. Conditional on $P_1$, covariance $(y_i, \bar{y}) = 1/n$. Thus conditional on $P_1$, $(y_i, \bar{y})$ is

$$\text{Normal} \left( \begin{pmatrix} P_1 \\ P_1 \end{pmatrix} , \begin{pmatrix} 1 & 1/n \\ 1/n & 1/n \end{pmatrix} \right).$$
Hence

\[ h_i(y_i, \bar{y} \mid P_1) = (2\pi)^{-1} \begin{vmatrix} 1/n & 1/n \\ 1/n & 1/n \end{vmatrix}^{-1/2} \times \exp \left\{ -\frac{1}{2} \left( y_i - \bar{y} \mid P_1 \right)^T \begin{pmatrix} 1 \ 1/n \\ 1/n \end{pmatrix}^{-1} \begin{pmatrix} y_i - P_1 \ y_i - P_1 \end{pmatrix} \right\} \]

\[ = (2\pi)^{-1} \frac{n}{\sqrt{n-1}} \exp \left\{ -\frac{1}{2} \frac{n}{n-1} \left[ (y_i - P_1)^2 - (\bar{y} - P_1)(y_i - P_1) \right. \right. \]

\[ \left. \left. + (P_1 - y_i)(\bar{y} - P_1) + n(\bar{y} - P_1)^2 \right] \right\}. \]

Define

\[ g_1(y_i, \bar{y}) = (2\pi)^{-1} \frac{n}{\sqrt{n-1}} \exp \left\{ -\frac{1}{2} \frac{n}{n-1} (y_i^2 - 2\bar{y}y_i) \right\}, \]

\[ g_2(\bar{y}, P_1) = \exp \left\{ -\frac{1}{2} \frac{n}{n-1} \left[ 2P_1 \bar{y} - P_1^2 + n(\bar{y} - P_1)^2 \right] \right\}. \tag{19} \]

Then \( h_i(y_i, \bar{y} \mid P_1) = g_1(y_i, \bar{y})g_2(\bar{y}, P_1). \) QED

We use Lemma 1 to prove Lemma 2 below. Lemma 2 states that if a trader is given \( \bar{y} \), then \( y_i \) provides no additional information about \( P_1 \) over that provided by \( \bar{y} \).

**Lemma 2.** Let \( m(P_1 \mid \bar{y}) \) be the density of \( P_1 \) conditional on \( \bar{y} \). Let \( \hat{m}(P_1 \mid \bar{y}, y_i) \) be the density of \( P_1 \) conditional on \( \bar{y} \) and \( y_i \). Then \( m(P_1 \mid \bar{y}) = \hat{m}(P_1 \mid \bar{y}, y_i) \) and hence \( E[\hat{P}_1 \mid \bar{y}] = E[\bar{P}_1 \mid \bar{y}, y_i] \) and \( \text{Var}[\hat{P}_1 \mid \bar{y}] = \text{Var}[\bar{P}_1 \mid \bar{y}, y_i] \).

**Proof.** By Bayes rule,

\[ \hat{m}(P_1 \mid \bar{y}, y_i) = \frac{g(P_1)h_i(y_i, \bar{y} \mid P_1)}{\int_{-\infty}^{\infty} g(P_1)h_i(y_i, \bar{y} \mid P_1) dP_1}, \tag{20} \]

where \( g(P_1) \) is the marginal density of \( \hat{P}_1 \).

\[ \hat{m}(P_1 \mid \bar{y}, y_i) = \frac{g(P_1)g_1(y_i, \bar{y})g_2(\bar{y}, P_1)}{\int_{-\infty}^{\infty} g(P_1)g_1(y_i, \bar{y})g_2(\bar{y}, P_1) dP_1} = \frac{g(P_1)g_2(\bar{y}, P_1)}{\int_{-\infty}^{\infty} g(P_1)g_2(\bar{y}, P_1) dP_1}. \tag{21} \]
The density of $\bar{y}$ given $P_1$, $f(\bar{y} \mid P_1)$, satisfies:

$$f(\bar{y} \mid P_1) = \int_{-\infty}^{\infty} h_i(y_i, \bar{y} \mid P_1) dy_i = \int_{-\infty}^{\infty} g_1(y_i, \bar{y}) g_2(\bar{y}, P_1) dy_i$$

$$= g_2(\bar{y}, P_1) \int_{-\infty}^{\infty} g_1(y_i, \bar{y}) dy_i. \quad (22)$$

The second equality in (22) follows from (18). By Bayes rule

$$m(P_1 \mid \bar{y}) = \frac{g(P_1) f(\bar{y} \mid P_1)}{\int_{-\infty}^{\infty} g(P_1) f(\bar{y} \mid P_1) dP_1} = \frac{g(P_1) g_2(\bar{y}, P_1) \int_{-\infty}^{\infty} g_1(y_i, \bar{y}) dy_i}{\int_{-\infty}^{\infty} g(P_1) g_2(\bar{y}, P_1) \left[ \int_{-\infty}^{\infty} g_1(y_i, \bar{y}) dy_i \right] dP_1}, \quad (23)$$

where the second equality follows from (22), $\int_{-\infty}^{\infty} g_1(y_i, \bar{y}) dy_i$ can be cancelled from the numerator and denominator in (23), hence

$$m(P_1 \mid \bar{y}) = \frac{g(P_1) g_2(\bar{y}, P_1)}{\int_{-\infty}^{\infty} g(P_1) g_2(\bar{y}, P_1) dP_1}. \quad (24)$$

Comparing (24) and (21), we see that $m(P_1 \mid \bar{y}) = \tilde{m}(P_1 \mid \bar{y}, y_i)$. QED

An immediate consequence of Lemma 2 is that if a trader is given $\bar{y}$ and $y_i$, then inferences about $P_1$ will be made independently of $y_i$. That is $y_i$ is extraneous information if $\bar{y}$ is known.

We now prove the main theorem. The proof uses the fact that if $\alpha_0$ and $\alpha_1 > 0$ are known constants, then the conditional distribution of $P_1$ given $\alpha_0 + \alpha_1 \bar{y}$, is the same as the conditional distribution of $P_1$ given $\bar{y}$.

**Proof.** We show that if $P_0^*(y) \equiv \alpha_0 + \alpha_1 \bar{y}$, where $\alpha_0$ and $\alpha_1$ are given in (16) and (17), then $P_0^*(y)$ satisfies (12) for all $y$. From Lemma 2, $E[\tilde{P}_1 \mid y_i, \bar{y}] = E[\tilde{P}_1 \mid \bar{y}]$ and $\text{Var}[\tilde{P}_1 \mid y_i, \bar{y}] = \text{Var}[\tilde{P}_1 \mid \bar{y}]$. Under the distribution assumptions given at the beginning of this section it can be shown that the conditional distribution of $\tilde{P}_1$ given $\bar{y}$ is normal with moments given by

$$E[\tilde{P}_1 \mid \bar{y}] = \frac{\tilde{P}_1}{1 + n\sigma^2} + \frac{n\sigma^2 y}{1 + n\sigma^2} \quad (25a)$$

$$\text{Var}[\tilde{P}_1 \mid \bar{y}] = \frac{\sigma^2}{1 + n\sigma^2} \quad (25b)$$
(see Degroot [1970], p. 167). That is, the posterior mean of $\tilde{P}_1$ is a weighted average of the prior mean $\bar{P}_1$ and the sample mean $\bar{y}$. Note that $E[\tilde{P}_1 | y_i, \alpha_0 + \alpha_1 \bar{y}] = E[\tilde{P}_1 | \bar{P}_1] = E[\tilde{P}_1 | \bar{y}]$ if $\alpha_1 > 0$. Hence

$$E[\tilde{P}_1 | y_i, \alpha_0 + \alpha_1 \bar{y}] = \frac{\bar{P}_1 + n \sigma^2 \bar{y}}{1 + n \sigma^2}. \tag{26}$$

Similarly,

$$\text{Var}[\tilde{P}_1 | y_i, \alpha_0 + \alpha_1 \bar{y}] = \text{Var}[\tilde{P}_1 | \bar{y}] = \frac{\sigma^2}{1 + n \sigma^2}. \tag{27}$$

Using (13), (14), (26), and (27)

$$\sum_{i=1}^{n} X_i^d[P_0^*, y_i] = \sum_{i=1}^{n} \left\{ \frac{\bar{P}_1 + n \sigma^2 \bar{y}}{1 + n \sigma^2} - \frac{(1 + r)(\alpha_0 + \alpha_1 \bar{y})}{a_i \sigma^2} \right\}. \tag{28}$$

Using the definitions of $\alpha_0$ and $\alpha_1$ given in (16) and (17), (28) becomes

$$\sum_{i=1}^{n} X_i^d[P_0^*, y_i] = \sum_{i=1}^{n} \left\{ \frac{\bar{P}_1 + n \sigma^2 \bar{y}}{1 + n \sigma^2} - \frac{(1 + r)\bar{X}}{\sigma^2} - \frac{\bar{P}_1}{(1 + r)(1 + n \sigma^2)} \sum_{i=1}^{n} \frac{1}{a_i} \frac{1}{a_i} \sum_{i=1}^{n} \frac{\bar{P}_1 \sum_{i=1}^{n} \frac{1}{a_i} - \sigma^2 \bar{X}}{1 + n \sigma^2} + \frac{n \sigma^2}{(1 + n \sigma^2)(1 + r) \bar{y}} \right\}. \tag{29}$$

The right hand side of (29) reduces to $\bar{X}$. Thus for all $\bar{y}$

$$\sum_{i=1}^{n} X_i^d[P_0^*, y_i] = \bar{X}. \quad \text{QED}$$

Thus, in equilibrium the current price summarizes all the information in the market. Each trader finds his own $y_i$ redundant. This creates strong disincentives for investment in information, since each trader could do as well by observing only
the spot price, as he could if he also purchased a \( y_i \). Note that perfect competition is assumed among the traders, so that the information of all type \( i \) traders taken together affects \( P_0 \). However each individual trader of type \( i \) assumes that his trading activity has no affect on \( P_0 \). Thus, when one type \( i \) trader stops getting information via \( y_i \), \( P_0 \) is not affected, and \( \bar{y} \) can still be deduced from \( P_0 \).

If it costs \( C > 0 \) dollars to become informed then equilibrium will not exist. Each trader of type \( i \) stops collecting information because the information in \( P_0^* \) is superior to \( y_i \) and free. Is there another equilibrium with fewer types of informed traders? No. Let there be \( m \) types of informed traders, then the price will be a linear function of \( \sum_{j=1}^{m} y_j \) and thus transmits all information to uninformed traders. Consider any given informed trader of type \( m \); he feels that he could stop paying \( C \) dollars and though he would no longer get the information \( y_m \), the price system reveals the superior information, \( \sum_{j=1}^{m-1} y_j \). Hence, it is not an equilibrium to have \( m \) types of informed traders. If no traders are informed, then (for sufficiently small cost of becoming informed) each trader would want to become informed because he gets no information for free via the price system. Hence, with information costs positive equilibrium does not exist. The key to the argument is that no matter how many types of informed traders there are, the price system perfectly aggregates their information and removes the incentive from a trader of a particular type to become informed. This is because traders are price takers and assume the price system is not affected by their actions.

The result that there exists a price which is a sufficient statistic is not robust. For example if the dimensionality of the price system (i.e., the number of commodities less one) is smaller than the dimensionality of the sufficient statistic, then the price function cannot reveal the sufficient statistic. However, Grossman [1975] shows that when prices do not symmetrize people's information, there is a private incentive to open new markets and thus increase the dimensionality of the price system. The rest of this section is devoted to the uniqueness of equilibrium and the notion of "noise." The following section discusses the welfare aspects of equilibrium.

We now show that if there are two equilibria then they must contain different information. If \( P_0^*(y) \) and \( P_0^{**}(y) \) are equilibria and they contain the same information, then there exists a strictly monotone function \( H(\cdot) \) such that \( P_0^{**}(y) = H(P_0^*(y)) \). Below we show that either \( H(\cdot) \) is the identity mapping (i.e., \( P_0^{**}(y) \equiv P_0^*(y) \)) or one of them is not an equilibrium.

**Theorem 2.** If \( P_0^*(y) \) is an equilibrium, and \( P_0^{**}(y) = H(P_0^*(y)) \), where \( H(\cdot) \) is a strictly monotone function which is not the identity mapping, then \( P_0^{**}(y) \) is not an equilibrium.

---

1. Another paradoxical aspect of markets where prices are sufficient statistics is that each trader's demand function is a function only of the price and not his own information. If all traders ignore their information how does the information get into the price? This point is strongly related to the fact that the demand function in (13) is not an ordinary demand function. It gives the demands of traders in equilibrium. In models where the price conveys information, there is no longer the classical separation between demand functions and equilibrium prices. Classically, demand functions can be derived independently of the distribution of equilibrium prices. Here this is no longer possible. See Grossman and Stiglitz [1976] for an elaboration of this point.
Efficiency of Competitive Stock Markets Where Trades have Diverse Information 583

Proof. \( E[\tilde{P}_1 | y_i, P_0^y(y)] = E[\tilde{P}_1 | y_i, P_0^{**}(y)] \) and \( \text{Var}[\tilde{P}_1 | y_i, P_0^y(y)] = \text{Var}[\tilde{P}_1 | y_i, P_0^{**}(y)] \) since \( P_0^y \) and \( P_0^{**} \) contain the same information (i.e., they generate the same \( \sigma \)-field). Let \( X_i^d[P_0^y, y_i] \) be as defined in (13). Then if \( y' \) is some realization of \( y \),

\[
\bar{X} = \sum_{i=1}^{n} X_i^d[P_0^y, y_i'] = \sum_{i=1}^{n} \frac{E[\tilde{P}_1 | y_i', P_0^y]}{a_i \text{Var}[\tilde{P}_1 | y_i', P_0^y]} - (1 + r)P_0^y(y')
\]

\[
= \sum_{i=1}^{n} \frac{E[\tilde{P}_1 | y_i', P_0^{**}]}{a_i \text{Var}[\tilde{P}_1 | y_i', P_0^{**}]} - (1 + r)P_0^{**}(y') \,. \tag{30}
\]

Assume, without loss of generality, that \( P_0^y(y') > P_0^{**}(y') > 0 \). Then

\[
\sum_{i=1}^{n} \frac{E[\tilde{P}_1 | y_i', P_0^{**}]}{a_i \text{Var}[\tilde{P}_1 | y_i', P_0^{**}]} - (1 + r)P_0^{**}(y') < \sum_{i=1}^{n} \frac{E[\tilde{P}_1 | y_i', P_0^y]}{a_i \text{Var}[\tilde{P}_1 | y_i', P_0^y]} - (1 + r)P_0^y(y') \,. \tag{31}
\]

The right hand side of (31) is just \( \sum_i X_i^d[P_0^{**}, y_i] \). Thus by (30), \( \sum_i X_i^d[P_0^{**}, y_i'] > \bar{X} \). (This clearly also holds in a non-degenerate neighborhood of \( y_i' \), as the prices are continuous functions of \( y \).) Thus \( P_0^{**} \) is not an equilibrium. QED

Thus there cannot be two equilibria with the same information content. The appendix shows that equilibria is unique in the class of all linear functions of \( y \). We do not know whether there are prices which are non-linear functions of \( y \) and are also equilibria.

The result that there is an equilibrium which is a sufficient statistic is not robust. It will not hold if there is no noise in the price system. (See Grossman [1975] for an elaboration of the notion of “noise.”) Suppose the total stock of the risky asset, \( x \), is unknown to all traders. Suppose that they have a common prior distribution on \( \tilde{x} \), such that \( \tilde{x} \) is independent of \( \tilde{P}_1 \) and \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \). When \( x \) is random all traders will know that the price which clears the market depends not only on \( y \) but also on the realization of \( \tilde{x} \). Define an equilibrium as a mapping \( P_0(y, x) \), such that for all \( (y, x) \)

\[
\sum_{i=1}^{n} \frac{E[\tilde{P}_1 | y_i, P_0(y, x)] - (1 + r)P_0(y, x)}{a_i \text{Var}[\tilde{P}_1 | y_i, P_0(y, x)]} = x \,. \tag{32}
\]

Clearly an equilibrium \( P_0(y, x) \) cannot be a constant function of \( x \) (i.e., a function of \( y \) alone). This is because if \( P_0(y, x) \) is a function only of \( y \), then the left hand side of (32) does not depend on \( x \), while the right hand side of (32) does depend on \( x \). As \( x \) and \( y \) are independent this is impossible. Thus it will not be possible to infer \( \bar{y} \) from \( P_0(y, x) \) unless \( x \) has a degenerate distribution.
4. WELFARE ASPECTS OF EQUILIBRIUM AND CONCLUSIONS

Let $\bar{x}_{fi}$ be the $i$th trader's endowment of the risky asset. Define

$$u^*_i(X_{fi},x_i,y) \equiv E\left[U_i\left(\tilde{W}_{i1}\right) | \tilde{y}\right] = E\left[U_i((1+r)X_{fi} + \tilde{P}_i x_i) | \tilde{y}\right].$$

(33)

Consider the pure exchange economy where traders have utility function $u^*_i(\cdot)$ and endowments are $(\bar{x}_f, \bar{x}_r)$. A competitive equilibrium for such an economy is Pareto efficient. The equilibrium $P^*_0(y)$ is an equilibrium for this economy because it gives each trader the information $\tilde{y}$, and this is equivalent to having $y$. The utility frontier of the central planner with information $\tilde{y}$ is equivalent to the utility frontier with information $y$, since $\tilde{y}$ is a sufficient statistic. Therefore the equilibrium $P^*_0(y)$ is efficient to the extent that it yields allocations which a central planner with all the information $y$ would choose as optimal. If there are any other equilibria they cannot yield more efficient allocations. Similarly the noisy equilibrium $P_d(y,x)$ cannot yield more efficient allocations.

The paradox we must face is that $P^*_0(y)$, by being so efficient, removes incentives for individuals to collect information. If information is costly then no individual will purchase it if he can observe $P^*_0$. Therefore $P^*_0(y)$ is not an equilibrium if information is costly. Only an imperfect information equilibrium can be an equilibrium in an economy where information is costly. There may be imperfect information equilibria. These equilibria would have a chance of persisting in an economy where information is costly.

Hayek ([1945], p. 527) has written:

"We must look at the price system as... a mechanism for communicating information if we want to"

2. Where $W_0 \equiv \bar{x}_{fi} + P^*_0(y) \bar{x}_f$.

3. We have shown that for each $y$ the central planner cannot dominate the competitive allocations of Section 3. However, if we consider a replicated economy the variance of $y$ will make life more risky in the competitive economy. A central planner could counteract this by equalizing allocations for a given individual across different realizations of $\tilde{y}$. Thus a central planner could achieve, for all $i$, higher $E u^*_i$ than the competitive market of Section 3 even though the planner could not improve $E[u^*_i | y]$ for each individual. This occurs because in Section 3 we have not given the competitive economy the ability to insure against the risks of variation in $W_0$ due to variation in $\tilde{y}$. If before traders observe $P_0$, we allow traders to trade promises to deliver income contingent on the realizations of $\tilde{y}$, then the competitive market will do as well as the central planner in allocating the risks associated with $\tilde{y}$. Once the market for risky assets opens the equilibrium price will be a sufficient statistic just as in Section 3.

4. Because of the strong portfolio separation property of the utility functions, the competitive equilibrium holdings of risky assets will be the same when all traders have the same beliefs irrespective of which beliefs they have, as long as $P_1$ is conditionally normally distributed. For a given $y$, the equilibrium allocations generated by $P^*_0$, strictly Pareto dominate the allocations generated in the naive economy where people observe only $y_i$ and prices are given by $P_1(y)$ in (13a). However, the allocations generated by $P^*_0$ do not Pareto dominate the allocations generated by the competitive equilibrium where all traders ignore $y_i$ and use only their prior distribution on $P_1$. This result that no information is as good as all the information is a peculiarity of utility functions which have the strong portfolio separation property, and is of little interest. It will always be true that when prices are sufficient statistics the central planner with all information will not be able to Pareto dominate the competitive allocations conditional on $y$. 
understand its real function... The most significant fact about this system is the economy of knowledge with which it operates, or how little the individual participants need to know in order to be able to take the right action... by a kind of symbol, only the most essential information is passed on...”

In an economy with complete markets, the price system does act in such a way that individuals, observing only prices, and acting in self interest, generate allocations which are efficient. However, such economies need not be stable because prices are revealing so much information that incentives for the collection of information are removed. The price system can be maintained only when it is noisy enough so that traders who collect information can hide that information from other traders. When this occurs some traders want very much to know why prices are, for example, unusually high. It is not enough for traders to observe only prices.

REFERENCES


